Radicals of Rings and Related Topics
August 2–8, 2009
Warsaw, Poland

Are polynomial rings over nil rings Brown-McCoy radical?

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Based on a joint work
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August 4, 2009

Köthe's Conjecture

There are many equivalent statements for Köthe's problem (1930) to have a positive answer.

- The sum of two nil left ideals is a nil left ideal.
- The 2×2 matrix ring over a nil ring is nil. (Krempa 1972)
- If N is a nil ring, then $N[x]$ is Jacobson radical. (Krempa 1972)
- A ring which is the sum of a nil subring and a nilpotent subring must be nil. (Ferrero and Puczyłowski 1989)

Brown-McCoy radical

- [Puczyłowski and Smoktunowicz, 1998]
  Let N be a nil ring. Then $N[x]$ is Brown-McCoy radical.
  This means that $N[x]$ cannot be homomorphically mapped onto a ring with identity.
- [Smoktunowicz, 2003]
  Let R be a ring. If $R[x]$ is Jacobson radical, then $R[x, y]$ is Brown-McCoy radical.

Thus, if N is a nil ring such that $N[x, y]$ is not Brown-McCoy radical, then N is a *counterexample* to destroy the hope for a positive solution of Köthe's problem.

The question

Let N be a nil ring. Is $N[x, y]$ Brown-McCoy radical? Or can we find some homomorphic image of $N[x, y]$ which has an identity?

Here we claim that if N is nil such that $N[x, y]$ can be mapped homomorphically into some ring with identity, then N cannot has finite characteristic.

The Theorem

Theorem

If N is a nil ring with $pN = 0$ for some prime p, then $N[x, y]$ cannot be homomorphically mapped onto a ring with identity.

Theorem

If N is a nil $\mathbb{Z}_p$-algebra, then $N[x, y]$ is Brown-McCoy radical.

Theorem (Puczyłowski and Smoktunowicz, 1998)

The polynomial ring $R[x]$ over a ring R is Brown-McCoy radical if, and only if, R cannot be homomorphically mapped onto a ring containing non-nilpotent central elements.

The case $p = 2$

Let P be a ring with some non-nilpotent elements in its center Z.

Let $\varphi : N[x] \to P$ be an epimorphism.

Let $u = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ such that $\varphi(u) \in Z$, and is non-nilpotent.

Put $u_0 = a_0 + a_2x^2 + \cdots$ and $u_1 = a_1x + a_3x^3 + \cdots$.

So $u = u_0 + u_1$. (Just put $a_k = 0$ for $k > n$.)

Then $\varphi(u_0) + \varphi(u_1) = \varphi(u) \in Z$, and we have

$$\begin{aligned}\varphi(u)^2 &= (\varphi(u_0) + \varphi(u_1))^2 = \varphi(u_0)^2 + \varphi(u_1)^2 \\ &= \varphi(u_0^2) + \varphi(u_1^2) = \varphi(u_0^2 + u_1^2).\end{aligned}$$ [Explain]

Note the form of the polynomial $u' = u_0^2 + u_1^2$:

$$\begin{aligned} u' &= \left(a_0 + a_2x^2 + \cdots\right)^2 + \left(a_1x + a_3x^3 + \cdots\right)^2 \\ &= a_0^2 + (a_0a_2 + a_1^2 + a_2a_0)x^2 + \cdots \\ &\qquad + \left(\sum_{i=0}^{2k} a_ia_{2k-i}\right)x^{2k} + \cdots + a_n^2x^{2n}. \end{aligned}$$

Let $x_1 = x^2$ and $a'_k = \sum_{i=0}^{2k} a_ia_{2k-i}$. Then

$$u' = a'_0 + a'_1x_1 + \cdots + a'_kx_1^k + \cdots + a'_nx_1^n.$$

Remember that $a'_n = a_n^2$.

$$u' = a'_0 + a'_1 x_1 + \cdots + a'_k x_1^k + \cdots + a'_n x_1^n \quad \text{with} \quad a'_n = a_n^2$$

Put $u'_0 = a'_0 + a'_2 x_1^2 + \ldots$ and $u'_1 = a'_1 x_1 + a'_3 x_1^3 + \ldots$.

So $u' = u'_0 + u'_1$. (Again, put $a'_k = 0$ for $k > n$.)

Now, $\varphi(u'_0) + \varphi(u'_1) = \varphi(u') = \varphi(u)^2 \in Z$ is non-nilpotent. So

$$\varphi({u'}^2) = \varphi(u')^2 = \varphi(u'_0)^2 + \varphi(u'_1)^2 = \varphi({u'_0}^2 + {u'_1}^2).$$

Again, $u'' = {u'_0}^2 + {u'_1}^2$ is a polynomial of the form

$$u'' = {a'_0}^2 + (a'_0 a'_2 + {a'_1}^2 + a'_2 a'_0) x_1^2 + \cdots + \left(\sum_{i=0}^{2k} a'_i a'_{2k-i} \right) x_1^{2k} + \cdots + {a'_n}^2 x_1^{2n}.$$

Denote $x_2 = x_1^2 = x^{2^2}$ and $a''_k = \sum_{i=0}^{2k} a'_i a'_{2k-i}$. We get

$$u'' = a''_0 + a''_1 x_2 + \cdots + a''_k x_2^k + \cdots + a''_n x_2^n.$$

$$u'' = a_0'' + a_1'' x_2 + \cdots + a_k'' x_2^k + \cdots + a_n'' x_2^n \quad \text{with} \quad a_n'' = {a_n'}^2 = a_n^{2^2}$$

Continuing in this manner ℓ times, we get a non-nilpotent element $\varphi(u^{(\ell)}) = \varphi(u)^{2^\ell} \in Z$, where $u^{(\ell)}$ is an element of the form

$$u^{(\ell)} = a_0^{(\ell)} + a_1^{(\ell)} x_\ell + \cdots + a_k^{(\ell)} x_\ell^k + \cdots + a_n^{(\ell)} x_\ell^n,$$

with $x_\ell = x^{2^\ell}$ and $a_n^{(\ell)} = a_n^{2^\ell}$.

When ℓ is large enough, $a_n^{(\ell)} = 0$ since N is nil, and $u^{(\ell)}$ has at least one term less than u.

We can continue in this manner, and for sufficiently large m, we would have $u^{(m)} = 0$, and $\varphi(u^{(m)})$ is a non-nilpotent central element in P, a contradiction.

Hence the theorem holds for $p = 2$. [Explain for general case]

The general case

Here is the key lemma to the general case.

Lemma

Let A be a $\mathbb{Z}_p$-algebra and $b_0, b_1, \cdots, b_n \in A$ such that $b = b_0 + b_1 + \cdots + b_n$ is a central element of A. Then $b^p = \sum_{i_0+i_1+\cdots+i_p \equiv 0 \pmod p} b_{i_1} b_{i_1} \ldots b_{i_p}$.

Proof. For $1 \leq i \leq p-1$, let $a_i = \sum_{j \equiv i \pmod p} b_j$.
Then $b = a_0 + a_1 + \cdots a_{p-1}$, and it suffices to show that
$b^p = \sum_{i_1+i_2+\cdots+i_p \equiv 0 \pmod p} a_{i_1} a_{i_2} \cdots a_{i_p}$.

Regard A as a $\mathbb{Z}_p$-subalgebra of $A^{\#}[G]$, where $A^{\#}$ is A with identity adjoined, and $G=\langle g\rangle$ is the cyclic group of order p.

Let $u=a_0+a_1g+\cdots+a_{p-1}g^{p-1}$. Then $u^p=c_0+c_1g+\cdots+c_{p-1}g^{p-1}$ where each c_j is the sum of elements of the form $a_{i_1}a_{i_2}\cdots a_{i_p}$ with $i_1+i_2+\cdots+i_p\equiv j \pmod{p}$.

Let

$$v=b-u=a_1(1-g)+\cdots+a_{p-1}(1-g^{p-1})=w(1-g),$$

where

$$w=a_1+a_2(1+g)+\cdots+a_{p-1}(1+g+\cdots+g^{p-2}).$$

Since $1 - g$ is a central element of $A^{\#}[G]$, we have

$$v^p = (w(1-g))^p = w^p(1-g)^p = w^p(1-g^p) = 0.$$

As $u = b - v$ and b is a central element of $A^{\#}[G]$, we have

$$u^p = (b-v)^p = b^p - v^p = b^p \in A.$$

We get $c_0 + c_1 g + \cdots + c_{p-1}g^{p-1} = u^p$ is an element in A,

and so, $c_1 = c_2 = \cdots = c_{p-1} = 0$.

Now, $b^p = c_0 = \sum_{i_1+i_2+\cdots+i_p \equiv 0 \pmod{p}} a_{i_1} a_{i_2} \cdots a_{i_p}$ as claimed.

Here is the theorem to be proved.

Theorem

Let N be a nil $\mathbb{Z}_p$-algebra. Then $N[x]$ cannot be homomorphically mapped onto a ring with non-nilpotent central elements.

Proof. Assume the contrary. Let n be the smallest integer such that some polynomial $u = a_0 + a_1x + \cdots + a_nx^n \in N[x]$ of degree n can be mapped via a homomorphism onto a non-nilpotent central element of some ring. Also assume that the nilpotency index k of a_n is the smallest among such polynomials.

Let $f : N[x] \to P$ be such a homomorphism, where P is a ring with non-nilpotent central element.

Set $b = f(u)$ and $b_i = f(a_i x^i)$ for $i = 0, 1, \cdots, n$.

Then $b = b_0 + b_1 + \cdots + b_n$ is a central element of P,

and so $b^p = f(v)$, where

$$v = \sum_{i_1+i_2+\cdots+i_p \equiv 0 \pmod p} a_{i_1} a_{i_2} \cdots a_{i_p} x^{i_1+i_2+\cdots+i_p}.$$

Write $v = c_0 + c_1 x^p + c_2 x^{2p} + \cdots + c_n x^{np}$, $c_i \in N$.

Set $w = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$.

Note that $c_n = a_n^p$.

Define $g : N[x] \to N[x]; r_0 + r_1 x + \cdots r_k x^k \mapsto r_0 + r_1 x^p + \cdots r_k x^{pk}$.

Then $g(w) = v$.

Now $f \circ g : N[x] \to P$ is a homomorphism.

Also, w is a polynomial in $N[x]$ of degree n such that $f(g(w)) = f(v) = b^p$ is a non-nilpotent central element of P.

However, the nilpotency index of $c_n = a_n^p$ is $\leq [k/p] + 1 < k$, a contradiction. This completes the proof.

Questions

1. Maybe it is easier to solve Köthe's problem in finite characteristic?
2. Is $N[x, y, z]$ Brown-McCoy radical when N is a nil ring?
3. How about more variables?