On some classes of finite groups

(notes to the talk)

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All groups considered in this talk are finite. The radicals here are understood in the sense of Kurosh and Amitsur, but other radicals will turn up from time to time. The set of all primes is denoted by $\mathbb{P}$, p will always denote a prime and π will always denote some subset of $\mathbb{P}$. Moreover $\mathcal{S}$ denotes the class of all finite simple groups, $\mathcal{F}$ denotes the class of all finite groups and $\emptyset$ denotes the empty class of groups.

Recall that a subgroup H of a group G is said to be *subnormal* in G if there exists a chain

$$G = G_0 \geqslant G_1 \geqslant G_2 \geqslant \cdots \geqslant G_r = H \geqslant 1$$

of subgroups of G with G_i a normal subgroup of G_{i-1} (for $i = 1, \ldots, r$). If H is subnormal in G, we shall write $H \operatorname{sn} G$. The series itself is called a *subnormal series* and G_{i-1}/G_i are called the *factors* of this series. If all the factors of a normal series are simple, then the series is called a *composition series* and the factors are called *composition factors* of G. Any finite group G possesses at least one composition series. The *Jordan-Hölder* theorem states that in a finite group any two composition series have the same length and there is a bijection between the two sets of factors in isomorphic pairs. Therefore these simple factors are uniquely determined by G (apart from ordering).

The history of radicals is well-known to the participants of this meeting. Let me only recall that between 1952 and 1954 Amitsur and Kurosh [1, 2, 3, 11] defined the notion of radical classes and proved basic results concerning them.

The starting point of the theory of classes of groups is the attempt to develop a generalized Sylow theory, which leads to an investigation into the problem of the existence of certain conjugacy classes of subgroups in finite groups.

Perphaps the most well-known existence and conjugacy theorem is Sylow's theorem which says, in its simplest form, that if p is a prime and G is a group, then the maximal p-subgroups of G are conjugate in G.

Theorem 1. (Sylow, 1872) *Let G be a group and p a prime. Then the maximal p-subgroups of G are conjugate in G.*

The beginning of this particular area of finite group theory came with P. Hall's generalization of Sylow's theorem to solvable groups.

It shows that, in a solvable group G, and corresponding to any set π of primes, there is a unique conjugacy class of subgroups of G whose orders involve only primes in π and whose indices involve no primes in π the so-called Hall π-subgroups.

Theorem 2. (P. Hall, 1928) *Let G be a solvable group and π any set of primes. Then the maximal π-subgroups of G are conjugate in G.*

We shall use $\mathrm{Hall}_\pi(G)$ to denote the set of all Hall π-subgroups of G.

By considering the order and index of Hall π-subgroups, it is easy to see that they satisfy the following three conditions.

Let N be a normal subgroup of a solvable group G. Then

1. $\mathrm{Hall}_\pi(G/N) = \{SN/N : S \in \mathrm{Hall}_\pi(G)\}$
2. $\mathrm{Hall}_\pi(N) = \{N \cap N : S \in \mathrm{Hall}_\pi(G)\}$
3. If $T/N \in \mathrm{Hall}_\pi(G/N)$ and $S \in \mathrm{Hall}_\pi(T),$ then $S \in \mathrm{Hall}_\pi(G)$.

In particular, Hall π-subgroups behave well as we pass from G to a factor group G/N or to a normal subgroup N. It is these three properties that have led to wide generalizations, the first and third properties leading to the theory of saturated formations and the associated projectors and the second property to the theory of Fitting classes and injectors.

The first evidence that further extensions were possible came in 1961 with the following discovery of R.W. Carter: every solvable group has self-normalizing nilpotent subgroups (or *Carter subgroups* as they became known) and these form a single conjugacy class of the group.

Recall that a group G is said to be *nilpotent* if every subgroup of G is subnormal.

A *Carter subgroup* of a group G is a self-normalizing nilpotent subgroup of G.

Theorem 3. (R.W. Carter, 1961) *A solvable group G has a Carter subgroup and any two Carter subgroups of G are conjugate in G.*

It is clear that a Carter subgroup of a group G is a maximal nilpotent subgroup of G. However, if G is a non-nilpotent solvable group, then G has a maximal nilpotent subgroup which is not a Carter subgroup.

W. Gaschütz viewed the Carter subgroups as analogues of the Sylow and Hall subgroups of a solvable groups and in 1963 published a seminar paper where a broad extension of the Hall and Carter subgroups was presented. The theory of formations was born. The new subgroups had many of properties of Sylow and Hall subgroups, but the theory was not arithmetic one, based on the orders of subgroups. Instead, the important idea was concerned with group classes having the same properties.

A formation is a class of groups that is closed under taking quotient images and subdirect products.

Definition 4. A class $\mathfrak{X}$ of groups is called a *formation* if it has the following two properties:

(1) every quotient group of every $\mathfrak{X}$-group is an $\mathfrak{X}$-group; (if $G \in \mathfrak{X}$ and $N \leqslant G$, then $G/N \in \mathfrak{X}$),

(2) if $N_1, N_2 \trianglelefteq G$ with $N_1 \cap N_2 = 1$ and $G/N_i \in \mathfrak{X}$ for $i = 1, 2$, then $G \in \mathfrak{X}$.

Examples of formations are the class of abelian groups, the class of nilpotent groups and the class of solvable groups.

Definition 5. A formation $\mathfrak{X}$ is said to be *saturated* if, whenever G is a group such that $G/\Phi(G)$ is an $\mathfrak{X}$-group, G is itself an $\mathfrak{X}$-group.

Recall that the *Frattini subgroup*, $\Phi(G)$, of a group G is the intersection of all maximal proper subgroups of G (and $\Phi(G) = G$ if there are no maximal proper subgroups of G). The Frattini subgroup can be considered as an analogue of Jacobson radical for modules.

For example, the formations of π-groups, nilpotent groups and solvable groups are saturated but the formation of abelian groups is not saturated.

Gaschütz proved that for any saturated formation of solvable groups $\mathfrak{X}$, any solvable group G possesses $\mathfrak{X}$-groups with certain special properties and these subgroups form a single conjugacy class. The subgroups in question are now called $\mathfrak{X}$-projectors of G, and (by a result of T.O. Hawkes) are characterized by the following property: a subgroup H of G is an $\mathfrak{X}$-projector of G if and only if, whenever $K \trianglelefteq G$, HK/K is a maximal $\mathfrak{X}$-subgroup of G/K. When $\mathfrak{X}$ is the saturated formation of solvable π-groups, the $\mathfrak{X}$-projectors of G are the Hall π-subgroups of G; and when $\mathfrak{X}$ is the saturated formation of nilpotent groups, the $\mathfrak{X}$-projectors of G are the Carter subgroups of G.

The theory of Fitting classes began when B. Fischer in his habilitation thesis wanted to see how far it is possible to dualise the theory of saturated formations and projectors by interchanging the roles of normal subgroups and quotient groups. A Fitting class should be regarded as the dual of a formation.

A Fitting class is a class of groups closed under taking normal subgroups and normal products of its members.

Definition 6. A class $\mathfrak{X}$ of groups is called a *Fitting class* if it has the following two properties:

(1) every normal subgroup of every $\mathfrak{X}$-group is an $\mathfrak{X}$-group;

(2) whenever a group G has normal $\mathfrak{X}$-subgroups N_1 and N_2 such that $G = N_1N_2$, then $G \in \mathfrak{X}$.

The class of π-groups is a Fitting class, and so is the class of nilpotent groups.

As it turn out, the definition of projector is the right thing to dualise in order to guarantee conjugacy. In 1967 the concept of injector appears in the celebrated paper by B. Fischer, W. Gaschütz and B. Hartley.

A subgroup V of G is said to be an $\mathfrak{X}$-*injector* of G if, whenever K is a subnormal subgroup of $G, V \cap K$ is a maximal $\mathfrak{X}$-subgroup of K. Fischer, Gaschütz and Hartley proved that a class of solvable groups $\mathfrak{X}$ is a Fitting class if and only if every solvable group has an $\mathfrak{X}$-injector. Moreover, the $\mathfrak{X}$-injectors form a single conjugacy class of subgroups of G. (Here no extra condition on the Fitting class corresponding to saturation for formations is needed).

When $\mathfrak{X}$ is the Fitting class of all solvable π-groups, the $\mathfrak{X}$-injectors of a solvable group, like its $\mathfrak{X}$-projectors, turn out to be the Hall π-subgroups. This is the only situation in which the injectors and projectors coincide, and so the two theories are quite independent generalizations of the classical Sylow and Hall subgroups.

These classes have been used to obtain a picture of the internal structure of groups. But there is a number of results which have something to say about the classes themselves. In investigating the classes above it is interesting to look at relations between them obtained by adding extra closure properties or by restricting the class of groups under consideration.

For example, in 1982, Bryce and Cossey proved the following remarkable fact.

Theorem 7. (R.A. Bryce and J. Cossey, 1982) *A subgroup closed Fitting class of solvable groups is a saturated formation.*

There are three volumes on classes of finite groups [5, 4, 8]:

1. *Finite Soluble Groups* by K. Doerk and T. Hawkes (1992),

2. *The theory of classes of groups* by Wenbin Guo (2000),

3. *Classes of finite groups.* by A. Ballester-Bolinches and L.M. Ezquerro (2006).

But what about radicals?

For a particular group G and class $\mathfrak{X}$ of groups, we may ask whether G has an $\mathfrak{X}$-*radical* and an $\mathfrak{X}$-*residual*; that is, whether G has a unique largest normal $\mathfrak{X}$-subgroup H (in which case H is the $\mathfrak{X}$-*radical* of G) and whether G has a unique smallest normal subgroup K such that G/K is an $\mathfrak{X}$-group (in which case K is the $\mathfrak{X}$-*residual* of G).

Theorem 8. *A class $\mathfrak{X}$ of groups is a formation if and only if it has the following two properties:*

(1) *every quotient group of every $\mathfrak{X}$-group is an $\mathfrak{X}$-group;*

(2) *every group has an $\mathfrak{X}$-residual.*

If G is a group and $\mathfrak{X}$ is a formation, then the $\mathfrak{X}$-residual is the intersection $\cap\langle N \trianglelefteq G \mid G/N \in \mathfrak{X}\rangle$.

Theorem 9. *A class $\mathfrak{X}$ of groups is a Fitting class if and only if it has the following two properties:*

(1) *every normal subgroup of every $\mathfrak{X}$-group is an $\mathfrak{X}$-group,*

(2) *every group has an $\mathfrak{X}$-radical.*

If G is a group and $\mathfrak{X}$ is a Fitting class, then the $\mathfrak{X}$-radical of G is the subgroup $\langle N \operatorname{sn} G \mid N \in \mathfrak{X}\rangle$.

Now we have the group versions of definitions of Hoehnke radical, complete radical, idempotent radical, Kurosh-Amitsur radical and Plotkin radical (see also [7]).

A radical γ may be defined as an assignment $\gamma : G \mapsto \gamma(G)$ designating a certain normal subgroup $\gamma(G)$ to each group G. Such an assignment γ is called a *Hoehnke radical*, if

i) $f\big(\gamma(G)\big) \subseteq \gamma\big(f(G)\big)$ for every epimorphism $f : G \twoheadrightarrow f(G)$,

ii) $\gamma\big(G/\gamma(G)\big) = 1$

for every group G. A Hoehnke radical γ may satisfy also the following conditions:

$$\begin{aligned} &\text{iii)} \quad \gamma \text{ is } \textit{complete}\text{: if } H \lhd G \text{ and} \\ &\qquad\ \ \gamma(H) = H \text{ then } H \subseteq \gamma(G); \\ &\text{iv)} \quad \gamma \text{ is } \textit{idempotent}\text{: } \gamma\big(\gamma(G)\big) = \gamma(G) \end{aligned}$$

for every group G.

It is known that γ is a Kurosh-Amitsur radical if and only if the assignment $G \mapsto \gamma(G)$ is a complete, idempotent Hoehnke radical.

A radical assignment $\gamma : G \mapsto \gamma(G)$ is called a *Plotkin radical*, if it satisfies conditions i), iii) and iv).

It can be seen that:

(1) if γ is a formation, then a rule which assigns to each group its γ-residual is a Hoehnke radical;

(2) if γ is a Fitting class, then a rule which assigns to each group its γ-radical is a complete idempotent radical;

(3) a rule which assigns to each group its solvable radical is a Kurosh-Amitsur radical, where the solvable radical of a group is the unique maximal solvable normal subgroup of the group.

In 1962 in his paper *Radicals in the theory of groups* [12] Kurosh showed that the general theory of radicals could be applied in the case of groups. Now we will concentrate on Kurosh-Amitsur radicals.

The definition of Kurosh-Amitsur radicals can be carried over almost word by word from the theory of rings to the group theory if we put the words *normal subgroup* instead of *ideal* and *group* instead of *ring.*

Definition 10. A class γ of groups is called a *radical class in the sense of Kurosh-Amitsur* (briefly a *radical*) if it has the following three properties:

(1) every quotient group of every γ-group is an γ-group;

(2) for every group G, the join $\gamma(G) = \langle H \lhd G \mid H \in \gamma \rangle$ is in γ;

(3) $\gamma\big(G/\gamma(G)\big) = 1$ for every group G.

$\gamma(G)$ is called the *γ-radical* of G. A group G is called a *γ-radical group* if $G \in \gamma$, that is $\gamma(G) = G$.

Theorem 11. *For any class γ of groups the following conditions are equivalent:*

(1) *γ is a radical class;*

(2) *γ is closed under taking quotient groups and under group extensions.*

Theorem 12. *For any class γ of groups the following conditions are equivalent:*

(1) *γ is a semisimple class;*

(2) *γ is closed under taking normal subgroups and under group extensions.*

The radical class (semisimple class) of all groups whose composition factors are isomorphic with a nonabelian simple group, say H, is not a Fitting class (formation) generated by H because the last one is the smallest class genenerated by H which is closed under taking direct product. The Fitting class (formation) of all nilpotent groups is not closed under group extensions, so it is not a semisimple class (radical class). The class of all solvable groups is a radical class, a semisimple class, a Fitting class and a formation.

Definition 13. A *semisimple radical class* is a class which is simultaneously a radical class and a semisimple class.

Theorem 14. *Let γ be a class of groups. Then the following conditions are equivalent:*

(1) *γ is a radical class which is closed with respect to taking normal subgroups;*

(2) *γ is a semisimple class which is closed with respect to taking quotient images;*

(3) *γ is a semisimple radical class;*

(4) *A group G belongs to γ if and only if all the composition factors of G belong to γ.*

The constructions of the lower radical and the upper radical for groups are similar to those for rings.

Example 15. [12] In the class of all groups the classes of the form $\mathcal{L}(\mathcal{M})$, where $\mathcal{M} \subseteq \mathcal{S}$, are the only radical classes, which are closed with respect to normal subgroups. In fact, if the radical class γ is closed with respect to normal subgroups then it contains all composition factors of any group belonging to it. Therefore $\mathcal{L}(\mathcal{M})$ is a semisimple radical class.

The analogous assertion is true for semisimple class too.

The wreath product of two groups is a useful tool for handling classes of groups.

Definition 16. The regular wreath product of two groups A and B which are assumed to be non-trivial, is defined as follows. Let F be the group of functions on B taking values in A with multiplication of $f, g \in F$ defined by

$$fg(x) = f(x)g(x) \text{ for all } x \in B.$$

Then F is the direct product of $|B|$ isomorphic copies of A.

If $f \in F$ and $b \in B$, we define $f^b \in F$ by

$$f^b(x) = f(xb^{-1}) \text{ for all } x \in B.$$

The group of automorphisms of F defined by

$$f \mapsto f^b \text{ for all } f \in F$$

is isomorphic to B and we shall identify it with B. The regular wreath product W of A and B is defined as the splitting extension of F by this group of automorphisms; that is, W is generated by B and F with the relations

$$b^{-1}fb = f^b \text{ for all } b \in B \text{ and } f \in F.$$

W is called *the regular wreath product of A by B*, written $A \wr B$. We shall refer to F as the *base group* of W. The subgroup of F consisting of all constant functions is the *diagonal subgroup*. It is clearly isomorphic to A. The centre of W coincides with the centre of the diagonal subgroup of W.

In the regular wreath product $A \wr B$ we shall need the following 'map' τ from F to A:

$$\tau(f) = \prod_{b \in B} f(b), \ f \in F.$$

Proposition 17. *[13] If B is not the trivial group, if $N \lhd W = A \wr B$ and if $N \cdot F = A \wr B$, then N contains $M = \{f \in F \mid \tau(f) \in A'\}$.*

Example 18. Let $\mathfrak{X}_1, \mathfrak{X}_2 \subseteq \mathcal{S}$, $\mathfrak{X}_1 \cap \mathfrak{X}_2 = \emptyset$, $\mathfrak{X}_1 \cup \mathfrak{X}_2 = \mathcal{S}$, $\mathfrak{X}_1 \neq \emptyset$, $\mathfrak{X}_2 \neq \emptyset$. We will show that $\mathcal{SL}(\mathfrak{X}_1) \neq \mathcal{L}(\mathfrak{X}_2)$ and $\mathcal{UL}(\mathfrak{X}_1) \neq \mathcal{L}(\mathfrak{X}_2)$. Assume that $G_1 \in \mathfrak{X}_1$ and $G_2 \in \mathfrak{X}_2$.
Assume that G_2 is nonabelian. We will show that $G_2 \wr G_1 \in \mathcal{SL}(\mathfrak{X}_1)$. Assume that $\mathcal{L}(\mathfrak{X}_1)(G_2 \wr G_1) \neq 1$. Then $1 \neq \mathcal{L}(\mathfrak{X}_1)(G_2 \wr G_1) \lhd G_2 \wr G_1$. Since $\mathcal{L}(\mathfrak{X}_1)(G_2 \wr G_1)$ is not contained in the base group F of $G_2 \wr G_1$ it follows that $F \cdot \mathcal{L}(\mathfrak{X}_1)(G_2 \wr G_1) = G_2 \wr G_1$. Proposition 17 now shows that $\mathcal{L}(\mathfrak{X}_1)(G_2 \wr G_1)$ contains

$$M = \{f \in F \mid \tau(f) \in G_2'\} = \{f \in F \mid \tau(f) \in G_2\} = F.$$

Hence $\mathcal{L}(\mathfrak{X}_1)(G_2 \wr G_1) = G_2 \wr G_1$, so $G_2 \wr G_1 \in \mathcal{L}(\mathfrak{X}_1)$, contrary to Example 15.
We thus get $\mathcal{SL}(\mathfrak{X}_1) \neq \mathcal{L}(\mathfrak{X}_2)$, so certainly $\mathcal{L}(\mathfrak{X}_2) \subsetneqq \mathcal{SL}(\mathfrak{X}_1)$.

Now assume that $\mathfrak{X}_2$ is a class of abelian simple groups. So G_2 is a cyclic group of order p, where $p \in \mathbb{P}$. We will show that $G_2 \wr G_1 \in \mathcal{SL}(\mathfrak{X}_1)$. Assume that $\mathcal{L}(\mathfrak{X}_1)(G_2 \wr G_1) \neq 1$. Then $1 \neq \mathcal{L}(\mathfrak{X}_1)(G_2 \wr G_1) \lhd G_2 \wr G_1$. Since $\mathcal{L}(\mathfrak{X}_1)(G_2 \wr G_1)$ is not contained in the base group F of $G_2 \wr G_1$ it follows that $F \cdot \mathcal{L}(\mathfrak{X}_1)(G_2 \wr G_1) = G_2 \wr G_1$. Proposition 17 now shows that $\mathcal{L}(\mathfrak{X}_1)(G_2 \wr G_1)$ contains

$$M = \{f \in F \mid \tau(f) \in G_2'\} = \{f \in F \mid \tau(f) = 1\}.$$

If $p \leqslant 5$, then we take $G_1 = A_5$. If $p > 5$, then we take $G_1 = A_p$. Then

$$N = \{f \in F \mid f(b) = f(1) \text{ for all } b \in G_1\} \subseteq M$$

and $G_2 \cong N \lhd G_2 \wr G_1$, so $G_2 \cong N \lhd \mathcal{L}(\mathfrak{X}_1)(G_2 \wr G_1)$. Then G_2 is a composition factor of $\mathcal{L}(\mathfrak{X}_1)(G_2 \wr G_1)$, contrary to Example 15.
We thus get $\mathcal{L}(\mathfrak{X}_1)(G_2 \wr G_1) = 1$. So $G_2 \wr G_1 \in \mathcal{SL}(\mathfrak{X}_1)$, and certainly $\mathcal{L}(\mathfrak{X}_2) \subsetneqq \mathcal{SL}(\mathfrak{X}_1)$.

Moreover, since $G_2 \wr G_1 \in \mathcal{SL}(\mathfrak{X}_1)$, if follows that $\mathcal{SL}(\mathfrak{X}_1) \neq \mathcal{UL}(\mathfrak{X}_1)$. In fact, $(G_2 \wr G_1)/F \cong G_1 \in \mathcal{L}(\mathfrak{X}_1)$

In general, for a radical class γ, $\gamma(A)$ does not always contain all subgroups of A which are in γ. We now consider radical classes for which this is the case.

Definition 19. (Kurosh) A radical class γ is *strict* if $B \leqslant \gamma(A)$ whenever $B \in \gamma$ and $B \leqslant A$.

Definition 20. A class σ is *strongly hereditary* if all subgroups of groups in σ are in σ.

Theorem 21. (Kurosh) *Let γ be a radical class with semisimple class σ. Then γ is strict if and only if σ is strongly hereditary.*

Theorem 22. (Kurosh) *Let $\mathcal{M}$ be a strongly hereditary class of groups. Then $\mathcal{U}(\mathcal{M})$ is a strict radical class.*

Theorem 23. *The only strict radical classes of groups closed respect to normal subgroups are $\{1\}$ and the class of all groups.*

Proof. Let $\gamma \neq \{1\}$ be a strict radical class closed respect to normal subgroups. If $1 \neq G \in \gamma$ then all the composition factors of G belong to γ. We will show that $\mathcal{S} \subseteq \gamma$.
Let G_1, G_2 be simple groups, $G_1 \in \gamma$ and $G_2 \notin \gamma$.
Assume that G_2 is nonabelian. We will show that $G_2 \wr G_1 \in \mathcal{S}\gamma$. Assume that $\gamma(G_2 \wr G_1) \neq 1$. Then $1 \neq \gamma(G_2 \wr G_1) \lhd G_2 \wr G_1$. Since $\gamma(G_2 \wr G_1)$ is not contained in the base group F of $G_2 \wr G_1$ it follows that $F \cdot \gamma(G_2 \wr G_1) = G_2 \wr G_1$. Proposition 17 now shows that $\gamma(G_2 \wr G_1)$ contains

$$M = \{f \in F \mid \tau(f) \in G_2'\} = \{f \in F \mid \tau(f) \in G_2\} = F.$$

Hence $\gamma(G_2 \wr G_1) = G_2 \wr G_1$, so $G_2 \wr G_1 \in \gamma$.
Since γ is closed respect to normal subgroups, it follows that $G_2 \in \gamma$, contrary to the assumption. So $\gamma(G_2 \wr G_1) = 1$ and $G_2 \wr G_1 \in \mathcal{S}\gamma$. Since $\mathcal{S}\gamma$ is strongly hereditary it follows that $G_1 \in \mathcal{S}\gamma$, contrary to the assumption.
Therefore if G is a nonabelian simple group then $G \in \gamma$.

Then G_2 is abelian and so G_2 is a cyclic group of order p, where $p \in \mathbb{P}$. We will show that $G_2 \wr G_1 \in \mathcal{S}\gamma$. Assume that $\gamma(G_2 \wr G_1) \neq 1$. Then $1 \neq \gamma(G_2 \wr G_1) \lhd G_2 \wr G_1$. Since $\gamma(G_2 \wr G_1)$ is not contained in the base group F of $G_2 \wr G_1$ it follows that $F \cdot \gamma(G_2 \wr G_1) = G_2 \wr G_1$. Proposition 17 now shows that $\gamma(G_2 \wr G_1)$ contains

$$M = \{f \in F \mid \tau(f) \in G_2'\} = \{f \in F \mid \tau(f) = 1\}.$$

It evident that M is a normal subgroup of $G_2 \wr G_1$, so $M \lhd \gamma(G_2 \wr G_1)$.
If $p \leqslant 5$, then we take $G_1 = A_5$. If $p > 5$, then we take $G_1 = A_p$. Then

$$N = \{f \in F \mid f(b) = f(1) \text{ for all } b \in G_1\} \subseteq M$$

and

$$G_2 \cong N \lhd G_2 \wr G_1, \text{ so } G_2 \cong N \lhd \gamma(G_2 \wr G_1).$$

Since γ is closed respect to normal subgroups it follows that $G_2 \in \gamma$, contrary to the assumption.
We thus get $\gamma(G_2 \wr G_1) = 1$. So $G_2 \wr G_1 \in \mathcal{S}\gamma$. Since $\mathcal{S}\gamma$ is strongly hereditary it follows that $G_1 \in \mathcal{S}\gamma_1$ contrary to the assumption.
Therefore $\mathfrak{S} \subseteq \gamma$ and by Example 15 $\gamma = \mathfrak{F}$. □

Remark. The theorem above was proved by Gardner in the situation when the universal class is the class of all groups. He used a free product of groups in the proof (see [6]).

Example 24. There are nontrivial strict radical classes of groups.

If γ is the class of solvable groups, then by Theorem 22 $\mathcal{U}(\gamma)$ is a strict radical class.

If γ is the class of p-groups, then $\mathcal{U}(\gamma)$ is a strict radical class.

Definition 25. A group G is *unequivocal* if $\gamma(G) = G$ or 1 for every radical class γ.

Theorem 26. *A group G is unequivocal if and only if all its composition factors are isomorphic.*

Problem 1. [12] (Kurosh) Give the complete description of all radicals in the class of all groups.

Problem 2. [12] (Kurosh) Give the complete description of all radicals in the class of finite groups.

Problem 3. (Kurosh-like) Give the complete description of all radicals in the class of finite solvable groups.

Problem 4. ([9] Ouestion 13.51) (A. N. Skiba) Is every finite modular lattice embeddable in the lattice of formations of finite groups?

We still know fairly little about the Fitting classes generated by arbitrary groups. Even the problem about the structure of the smallest Fitting class containing S_3 (symmetric group of degree three) still remains open [10].

Problem 5. Determine the smallest Fitting class containing S_3.

Definition 27. Let χ and $\mathfrak{X}$ be non-trivial classes Fitting classes of finite solvable groups such that $\chi \subseteq \mathfrak{X}$. Then χ is said to be normal in $\mathfrak{X}$ if an χ-injector of G is a normal subgroup of G for every $G \in \mathfrak{X}$.

In the paper [14] S. Reifferscheid established the existence of a unique maximal subgroup-closed Fitting class in which a given subgroup-closed Fitting class χ is normal. The dual problem, that is, the existence of a unique minimal subgroup-closed Fitting class being normal in a given subgroup-closed Fitting class $\mathfrak{X}$ is still open question.

Problem 6. Does there exist a unique minimal subgroup-closed Fitting class being normal in a given subgroup-closed Fitting class $\mathfrak{X}$?

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