# Multi-attribute Utility Functions

**Marek J. Drużdżel**

**Wydział Informatyki**

**Politechnika Białostocka**

*m.druzdzel@pb.edu.pl*

*http://aragorn.wi.pb.bialystok.pl/~druzdzel/*

# Outline

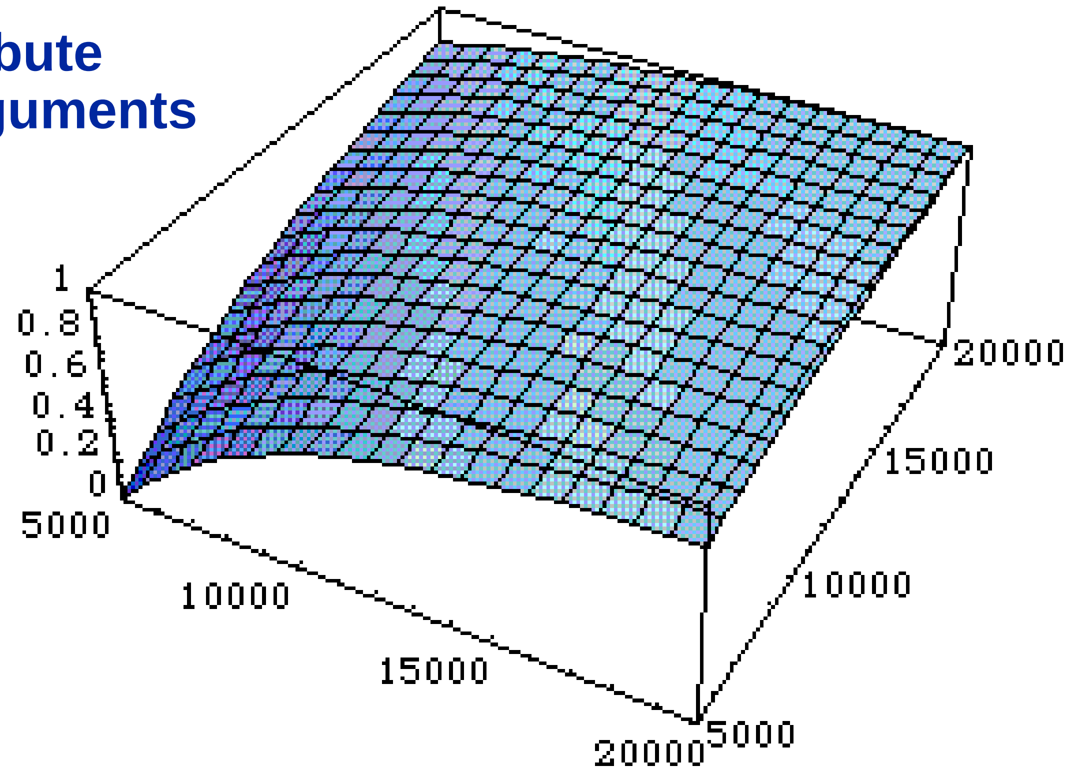
- The problem of multiple attributes
- Additive utility functions
- Assessing individual utility functions
- Assessing weights
- Some theory: Preferential, utility, and additive independences
- Multiplicative utility functions

# General Problem

# Multi-attribute utility

**When there are multiple attributes of a decision (quite typical ☺), we are facing a hard problem, a function of multiple arguments**

**Here is what a multi-attribute utility function of two arguments might look like.**




**Elicitation of a MAU function is hard (the number of points is exponential in the number of attributes).**

# Multi-attribute utility

An obvious solution is standardizing the shapes (similarly to canonical gates ☺)

Generally, simplifications along the lines of the following decomposition:

$$U(x_1, x_2, \ldots, x_n) = f(U_1(x_1), U_2(x_2), \ldots, U_n(x_n))$$

Solutions applied in practice:

- Additive linear function
- Multiplicative functions
- Risk tolerance-based functions

# Additive Linear Utility Functions

# Additive Utility Functions

- **Additive Utility Functions**
  - $U(x_1, x_2, \ldots, x_m) = k_1 U(x_1) + k_2 U(x_2) + \ldots + k_m U(x_m)$
  - Condition on weights: $k_1 + k_2 + \ldots + k_m = 1$
- **Additive Utility Functions are restrictive**
  - $U_i(x_i)$ may not exist, it may depend on values of other $x_j$
  - $U(x_1, x_2, \ldots, x_m)$ may not be a function of $U_i(x_i)$
  - $U(x_1, x_2, \ldots, x_m)$ may not be a linear combination of $U_i(x_i)$

# An Example

| | Portalo | Norushi | Standard Motors |
|---|---|---|---|
| Price ($1000) | 17 | 10 | 8 |
| Life Span (Years) | 12 | 9 | 6 |

# Assessing Individual Utility Functions

# Assessing Individual Utility Functions

- Proportional scores
- Ratios
- Standard utility function assessment

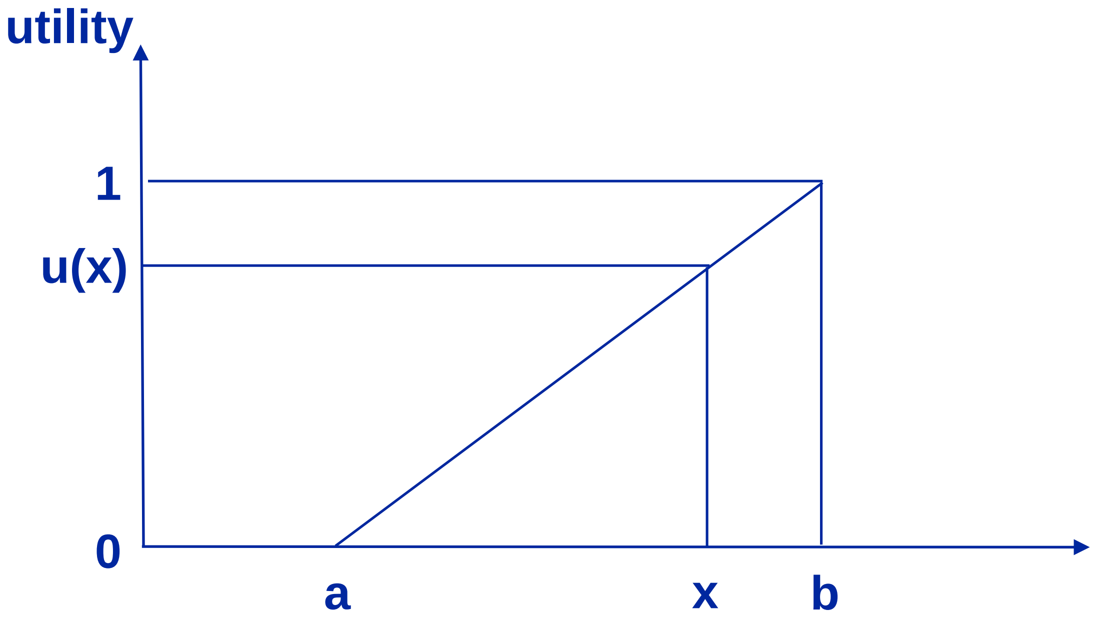
# Proportional Scores

- **Proportional score method requires that attributes have natural numerical measures**
- **They assume risk neutrality!**
- **Method**
  - **Set the utility value at worst and the best situation**
  - **Linearly interpolate utility value at points in between**

$$U(x) = \frac{x-a}{b-a}$$

# Proportional Scores

## Geometric view of the proportional score method



# Ratios

**Consider color of the car as an additional attribute.**

**Let blue be twice as good as red and yellow 2.5 times as good as red.**

**Let U''(red)=1, U''(blue)=2, and U''(yellow)=2.5, or alternatively U'(red)=30, U'(blue)=60, and U'(yellow)=75.**

**The only thing that remains is transforming these to the interval [0..1] (by a linear transformation!).**

**We have two equations with two unknowns:**

$$\begin{cases} 0=a+b*U'(red)=a+b*30 \\ 1=a+b*U'(yellow)=a+b*75 \end{cases}$$

**Solving these gives us a=-2/3 and b=1/45 and, effectively, U(blue)=-2/3+1/45*U'(blue)=-2/3+1/45*60=2/3**

# Assessing Weights

# Pricing out

- Choose a base attribute, usually represented in dollar amount
- Trading one attribute for another
- Example: (Clemen, page 547)
  - The decision maker may be indifferent between Standard Motors ($8,000, 6 years life span) and hypothetical car B ($8,600, 7 years life span)
  - An additional year of life span is worth $600/year
- Use proportional score method to calculate individual utilities and then solve for weights
- Keep in mind that weights should add up to 1

# Swing Weighting

- Use the worst or the best combination as benchmark (e.g., a car that will last for 6 years, costs $17K, and is red)
- Qualitative: rank hypothetical cars that have one attribute at the best value, the other attributes are at the worst value

| Attribute Swung from Worst to Best | Consequence to Compare | Rank | Rate | Weight |
|---|---|---|---|---|
| (Benchmark) | 6 years, $17,000, red | 4 | | |
| Life span | 12 years, $17,000, red | — | — | — |
| Price | 6 years, $8000, red | — | — | — |
| Color | 6 years, $17,000, yellow | — | — | — |

- Quantitative: assign the best 100, the worst (benchmark) 0, elicit the value for the other (hypothetical) cars

| Attribute Swung from Worst to Best | Consequence to Compare | Rank | Rate | Weight |
|---|---|---|---|---|
| (Benchmark) | 6 years, $17,000, red | 4 | 0 | |
| Life span | 12 years, $17,000, red | 2 | | |
| Price | 6 years, $8000, red | 1 | 100 | |
| Color | 6 years, $17,000, yellow | 3 | | |

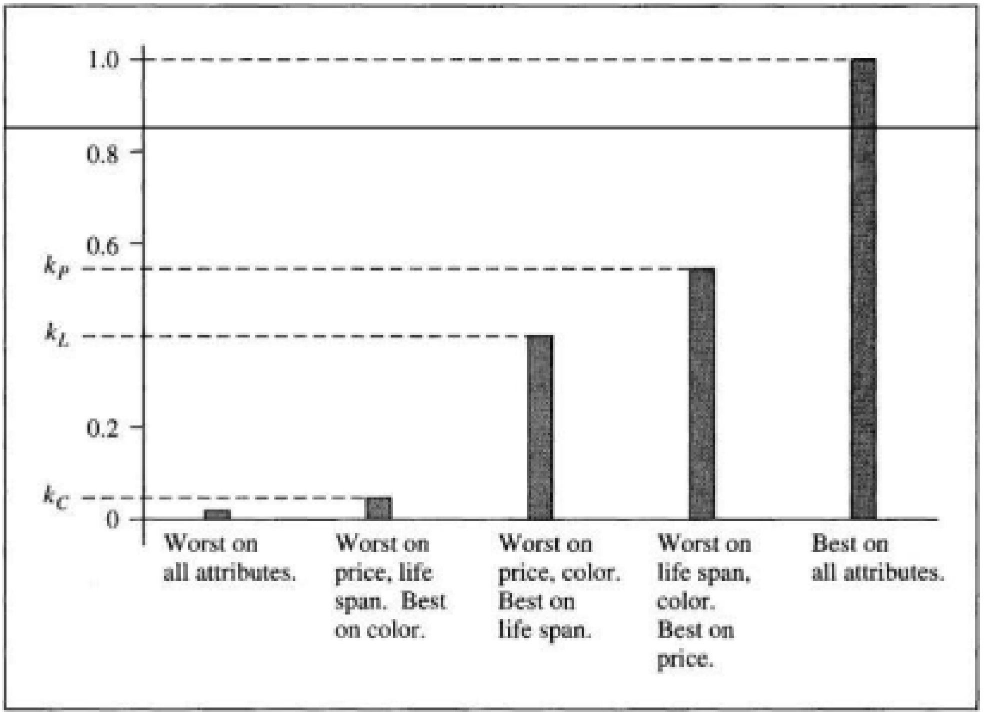
# Swing Weighting

- **Calculate weights by making sure that they add up to 1.0.**
- **Please note that this method rests on the property of the MAU function that individual utilities of worst outcomes are zero and utilities of the best outcomes are 1.0.**

| Attribute Swung from Worst to Best | Consequence to Compare | Rank | Rate | Weight |
|---|---|---|---|---|
| (Benchmark) | 6 years, \$17,000, red | 4 | 0 | |
| Life span | 12 years, \$17,000, red | 2 | 75 | 0.405 = 75/185 |
| Price | 6 years, \$8000, red | 1 | 100 | 0.541 = 100/185 |
| Color | 6 years, \$17,000, yellow | 3 | 10 | 0,054 = 10/185 |
| | | Total | 185 | 1.000 |

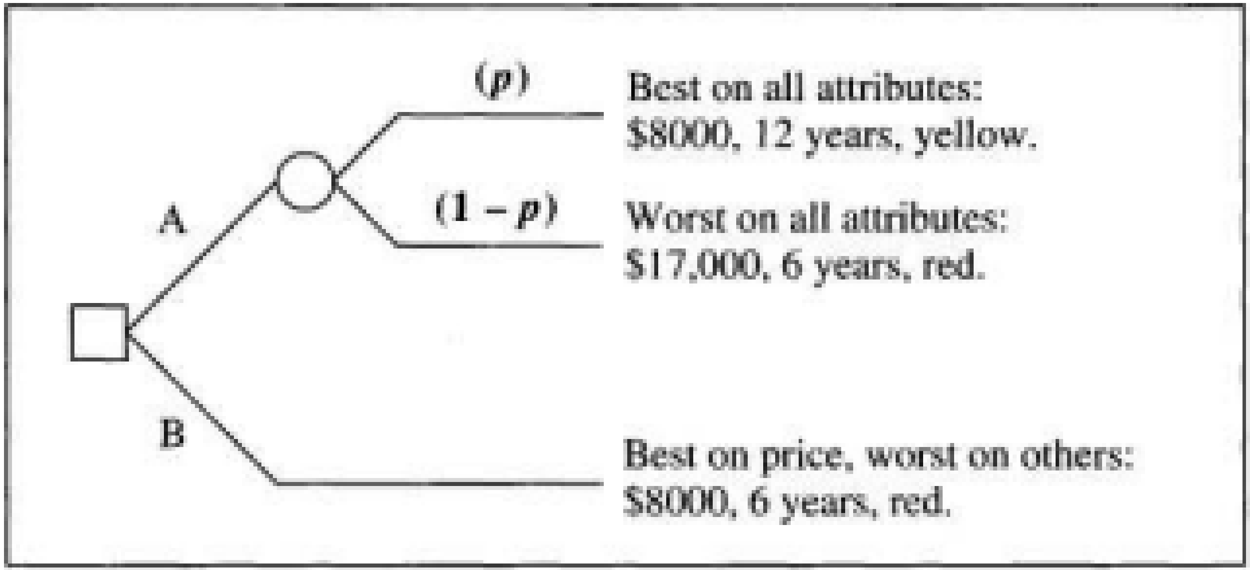
# Swing Weighting

- **Please note that this method rests on the property of the MAU function that individual utilities of worst outcomes are zero and utilities of the best outcomes are 1.0.**



# Lottery weights

- **Lottery has two choices**
  - best on one attribute, worst on the other
  - probability p of best on all
  - probability 1-p of worst on all
- **One more equation than necessary to solve for all weights. Can be used to check the validity of model ☺**



# Comparison of Weight Assessment Methods

- **Pricing out**
  - Attributes are naturally quantitative
  - Force thinking explicitly about tradeoffs
- **Swing weighting**
  - Questionable in estimating relative importance of attributes in numerical terms
- **Lottery weights**
  - Incorporates risk attitude well

# Some theory: Independencies

# Multi-attribute utility: Simplification of the problem

**Simplifications of the problem starts with a series of attribute independence tests:**

- **preferential independence**
- **utility independence**
- **additive independence**

# Preferential independence

**An attribute Y is said to be preferentially independent of X if preferences for specific outcomes of Y do not depend on the level of attribute X. In other words, the value of X does not influence our ordinal preferences for Y.**

**This condition is pretty intuitive and it holds most of the time.**

**Examples of violations?**

1. **The amount of homework and the course topic.**
2. **Car type and location.**

# Utility independence

**An attribute Y is considered utility independent of attribute X if preferences for uncertain choices involving different levels of Y are independent of the value of X. In other words, the value of X does not influence the certainty equivalent of a lottery involving Y.**

**Mutual utility independence: When the relation holds both ways.**

**Example when this is violated (from Keeney and Raiffa): Serious crime rates in two police precincts. The region's police chief does not want to appear as though he neglects one of the two precincts. An easy fix in that case is adding bonus to some values or transforming the function.**

# Multiplicative Utility Functions

# Implication of utility independence

**When mutual utility independence holds, we can write a two-attribute utility function as follows:**

$$U(x,y) = w_x U_x(x) + w_y U_y(y) + (1 - w_x - w_y) U_x(x) U_y(y)$$

**$U_x(x)$ and $U_y(y)$ are utility functions scaled to the interval [0,1], $w_x=U(x_1,y_0)$, $w_y=U(x_0,y_1)$.**

# Multiplicative form of multi-attribute utility

**This is known as the multiplicative form of a MAU function. It is a special functional form that gives a curvature in the utility function of multiple attributes and is capable of modeling such non-linearities as complements and substitutes.**

$$U(x,y) = w_x U_x(x) + w_y U_y(y) + (1 - w_x - w_y) U_x(x) U_y(y)$$

**The product term is what allows for modeling the interaction between the two attributes.**

# Complements and substitutes

$$U(x,y) = w_x U_x(x) + w_y U_y(y) + (1 - w_x - w_y) U_x(x) U_y(y)$$

The coefficient $(1 - w_x - w_y)$ can be interpreted quite nicely.

If positive, then higher values of both attributes at the same time will drive up the value of the utility function even higher (the attributes complement each other, e.g., two battles on one front, you need to win both, defeat on one is almost just as bad as defeat on both).

If negative, we are quite happy with having one or the other and don't necessarily need to have both (they substitute each other, e.g., two branches of a company, two investments).

# Utility independence

How do we demonstrate that this functional form implies mutual utility independence?

Take one value of y: The function will transform to the utility $U_x$, although it will be its linear transformation.

For another value of y, it will be another linear transformation.

The utility function for x will be exactly the same, because it is determined up to a linear transformation anyway.

How to go the other way, i.e., demonstrate that you need this functional form to have mutual utility independence?

Left as a homework exercise ☺.

# Additive independence

When $w_x + w_y = 1$, the multiplicative function simplifies to

$$U(x,y) = w_x U_x(x) + w_y U_y(y)$$

This is precisely when additive independence holds.

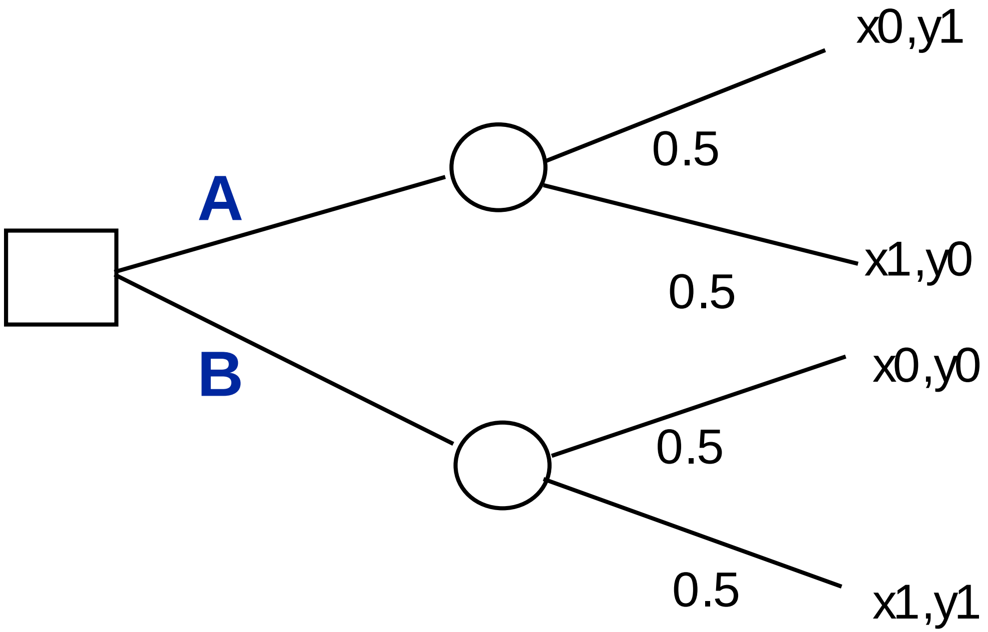
In general

- $U(x_1, x_2, \ldots, x_m) = k_1 U(x_1) + k_2 U(x_2) + \ldots + k_m U(x_m)$
- Constraint on weights: $k_1 + k_2 + \ldots + k_m = 1$

Additive linear utility function is quite often used and abused (used without checking whether it is a good approximation).

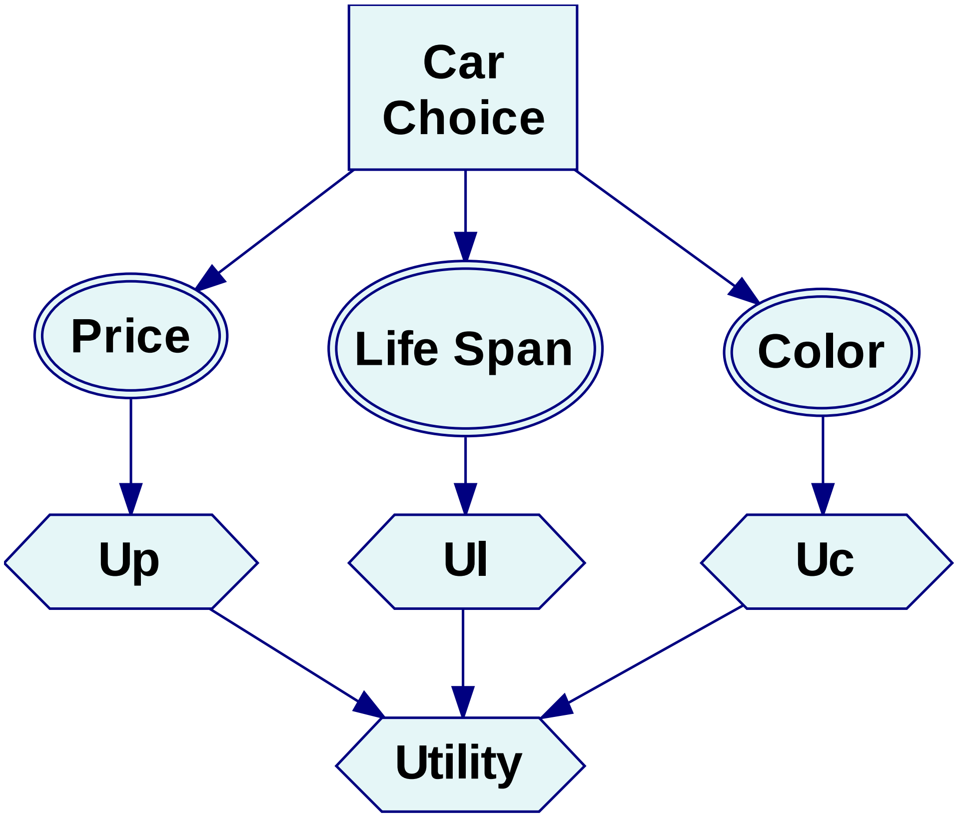
# Multi-attribute utility assessment

**Are you indifferent between the two choices? If so, then they are additively independent, but if you prefer one over the other, then they are not. A good example: service and reliability — most of us prefer when at least one of them is good to the situation when you can be screwed up on both or have both good.**



# Examples

# Car Choice: Model



# Oregon Library

# Making Hard Decisions

**An Introduction to Decision Analysis**

**2nd Edition**

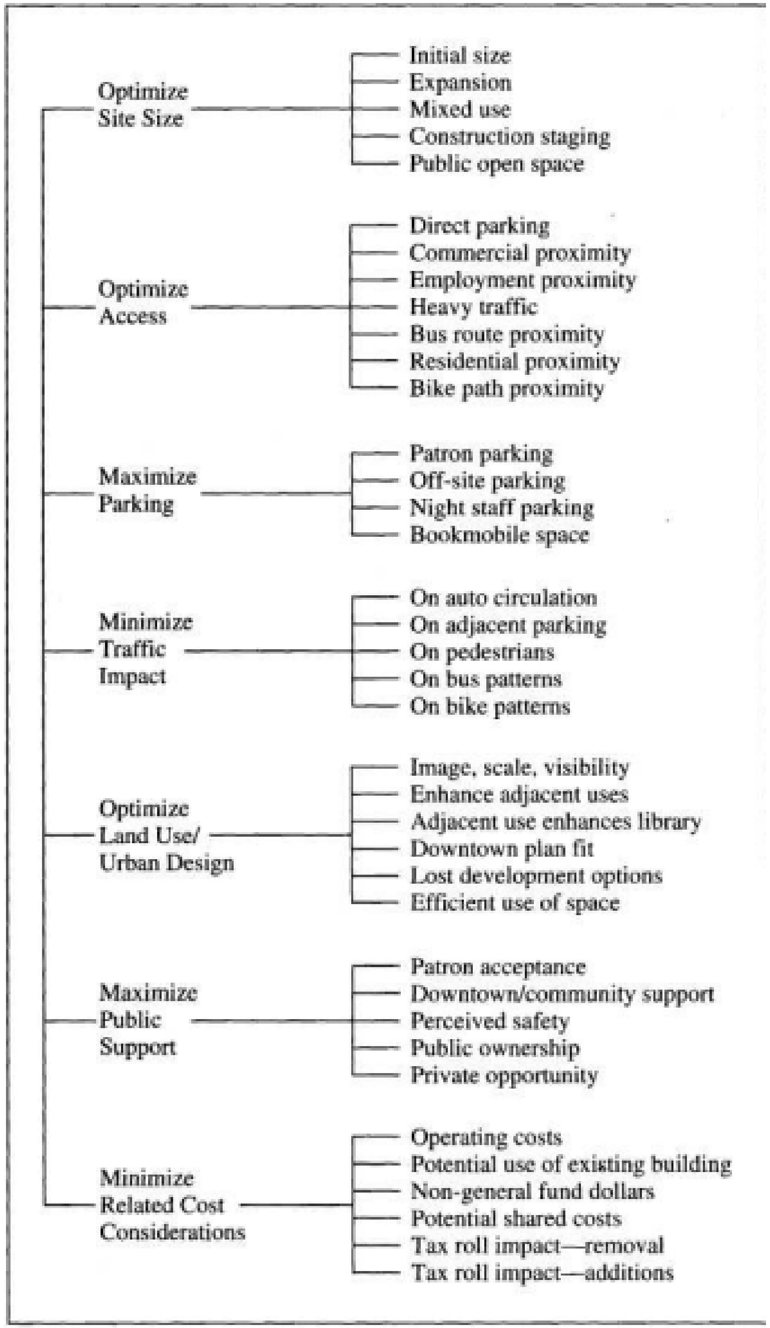
**Robert T. Clemen**

Fuqua School of Business
Duke University

Duxbury Press
An Imprint of Wadsworth Publishing Company
I(T)P® An International Thomson Publishing Company

Belmont • Albany • Bonn • Boston • Cincinnati • Detroit • London
Madrid • Melbourne • Mexico City • New York • Paris • San Francisco
Singapore • Tokyo • Toronto • Washington

# Oregon Library

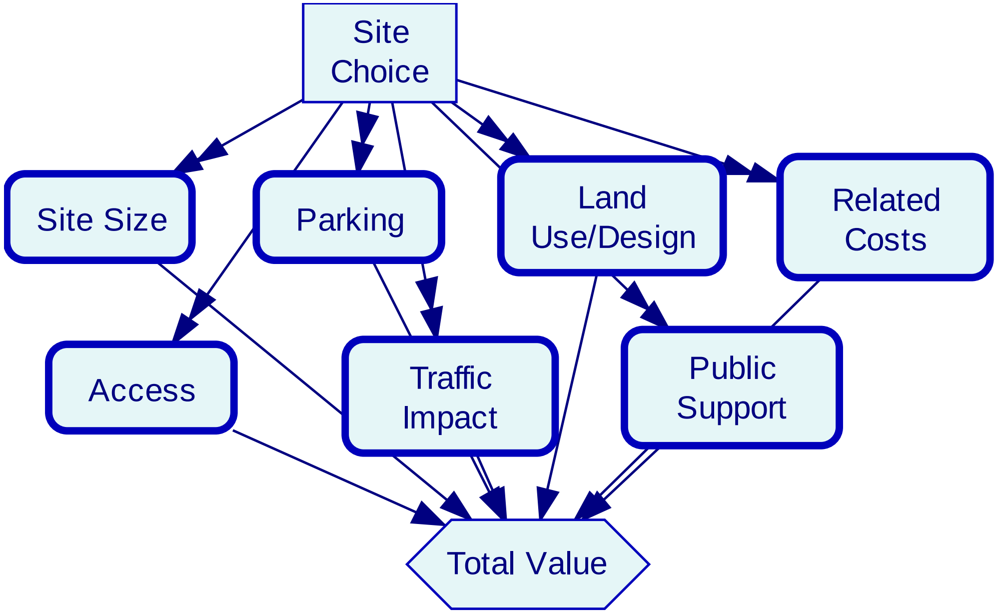


# Oregon Library

| Attributes | % | Site 1 | Site 2 | Site 3 | Site 4 |
|---|---|---|---|---|---|
| Public Support (19.0%) | | | | | |
| Patron Acceptance | 25 | 1.00 | 0.33 | 0.67 | 0.00 |
| DT/Community Support | 25 | 1.00 | 0.67 | 0.33 | 0.00 |
| Perceived Safety | 25 | 1.00 | 033 | 1.00 | 0,00 |
| Public Ownerhip | 17 | 0.00 | 1.00 | 1.00 | 0.00 |
| Private Opportunity | 8 | 1.00 | 0.00 | 1.00 | 1.00 |
| *Subtotals* | | *15.77* | *9.55* | *14.25* | *1.52* |
| Related Costs (21.1%) | | | | | |
| Operating Costs | 20 | 0.00 | 1.00 | 1.00 | 1.00 |
| Use of Existing Building | 20 | 1.00 | 0.00 | 0.00 | 0.00 |
| No General Fund $ | 30 | 0,00 | 1.00 | 1.00 | 1.00 |
| Tax Roll Impact, Removal | 10 | 0.00 | 1.00 | 1.00 | 0.00 |
| Tax Roll Impact, Added | 20 | 0.00 | 1.00 | 1.00 | 1.00 |
| *Subtotals* | | *4.22* | *16.88* | *16.88* | *14.77* |
| *Weighted Score* | | *45 70* | *55.51* | *70.22* | *52 78* |

| Attributes | % | Utilities Site 1 | Site 2 | Site 3 | Site 4 |
|---|---|---|---|---|---|
| Site Size (21.1%) | | | | | |
| Initial | 38 | 1.00 | 0.00 | 1.00 | 1.00 |
| Expansion (Horizontal) | 13 | 0.00 | 0.00 | 0.00 | 1.00 |
| Mixed Use | 25 | 0.00 | 1.00 | 1.00 | 1.00 |
| Construction Staging | 12 | 1.00 | 0.00 | 0.00 | 1.00 |
| Public Open Space | 12 | 1.00 | 0.00 | 0.00 | 0.00 |
| Subtotals | | 13.08 | 5.28 | 1329 | 18.57 |
| Access (20.6%) | | | | | |
| Direct Parking | 8 | 0.00 | 1.00 | 0.00 | 0.00 |
| Commercial Proximity | 23 | 0.00 | 1.00 | 0.67 | 1.00 |
| Employment Proximity | 15 | 0,50 | 1.00 | 0.00 | 1.00 |
| Heavy Traffic | 23 | 033 | 033 | 1.00 | 0.00 |
| Bus Route Proximity | 15 | 0.00 | 0.50 | 0.50 | 1.00 |
| Residential Proximity | 16 | 1.00 | 0.00 | 1.00 | 0.50 |
| Subtotals | | 6.40 | 12.55 | - 12,75 | 12.57 |
| Parking (53%) | | | | | |
| Patron Parking | 20 | 1.00 | 0,00 | 1.00 | 1.00 |
| Off-Site Parking | 60 | 0.00 | 1.00 | 0.33 | 0.33 |
| Bookmobile Parking | 20 | 1.00 | 0.00 | 1.00 | 1.00 |
| Subtotals | | 2.12 | 3.18 | 3.17 | 3.17 |
| Traffic Impacts (4.5%) | | | | | |
| Auto Circulation | 47 | 0.00 | 0.75 | 1.00 | 0.00 |
| Adjacent Parking | 29 | 0.00 | 0.00 | 1.00 | 0.00 |
| Bus Patterns | 24 | 1.00 | 1.00 | 1.00 | 0.00 |
| Subtotals | | 1.08 | 2.67 | 4.50 | 0.00 |
| Land Use/Design (8.4%) | | | | | |
| Image/Scale/Visibility | 13 | 0.00 | 1.00 | 0.00 | 0.00 |
| Enhance Adjacent Uses | 13 | 0.00 | 1.00 | 1.00 | 1.00 |
| Adj. Uses Enhance Lib* | 38 | 0.00 | 1.00 | 1.00 | 0.00 |
| Downtown Plan Fit | 13 | 1.00 | 0,00 | 1.00 | 1.00 |
| Lost Devel. Options | 23 | 1.00 | 0.00 | 0.00 | 0.00 |
| Subtotals | | 3.02 | 5.38 | 5.38 | 2.18 |

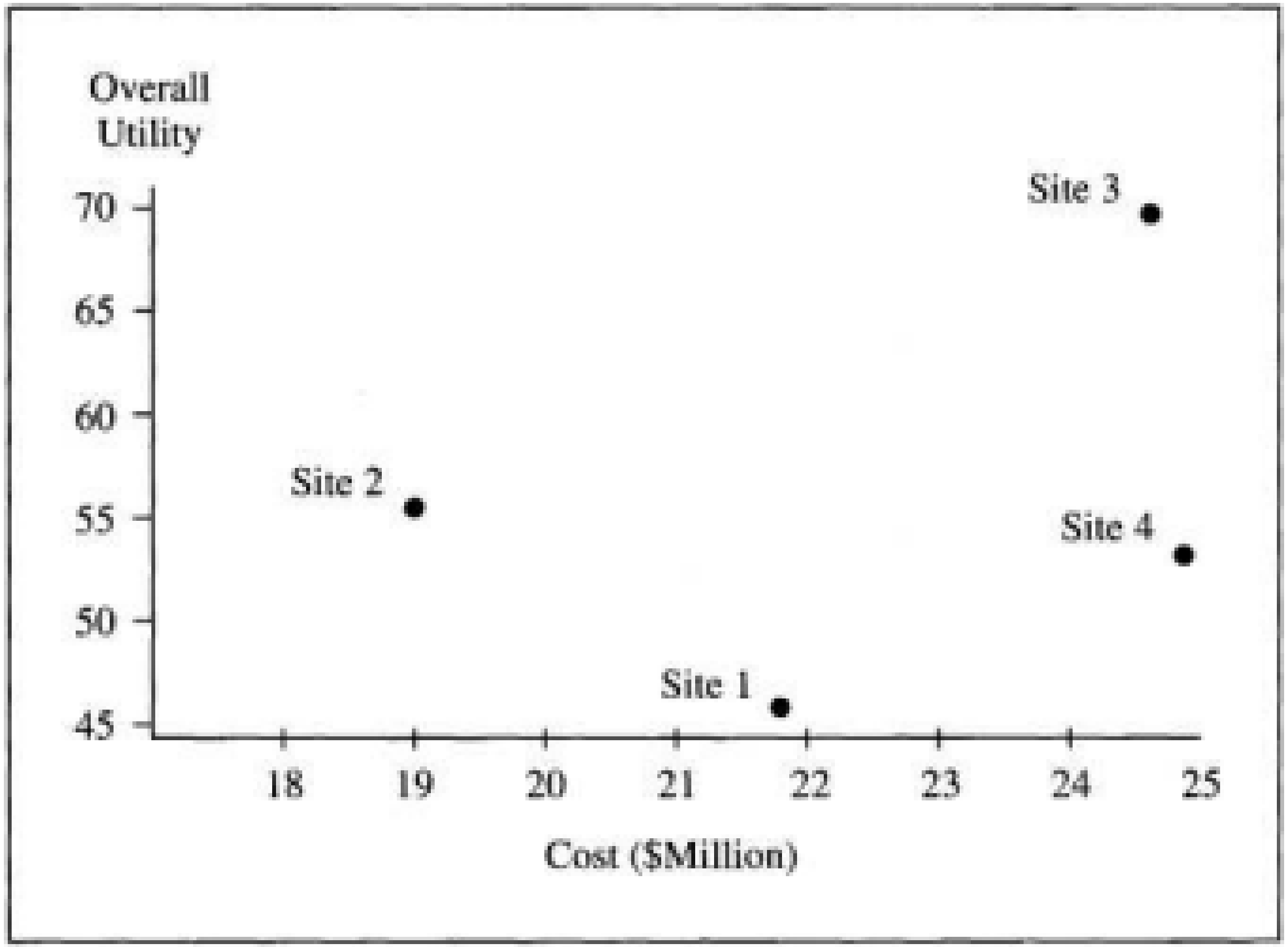
# Oregon Library: Model




**The seven submodels contain calculations of utilities of different attributes. Double-click on the submodel icon to examine the individual attribute calculations. Try navigating the model through the tree view as well!**

**Table 15.6: Matrix of weights and utilities for four library sites (this is actually an influence diagram equivalent to the table; as weights in the diagram run between 0 and 1, the final result in the node Total_Value is also in the interval between 0 and 1). Robert T. Clemen, Making Hard Decisions: An Introduction to Decision Analysis, Second Edition. Duxbury Press, 1996. The original source of the data is: Robertson, Sherwood and Architects (1987), Preliminary Draft Report: Eugene Public Library Selection Study. Executive Summary. Eugene, OR: Robertson/Sherwood.**

# Oregon Library: Dominance



# What If Everything Fails?

# MAU assessment: When everything fails

**What is mutual utility independence fails? You can always use direct assessment.**

**Sometimes transformations of the individual utility functions will work (e.g., instead of individual crime rates, take the average and difference between the two crime rates).**