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SKEW POWER SERIES RINGS OF DERIVATION TYPE

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ABSTRACT. In this paper, we contrast the structure of a noncommutative algebra R with that of the skew power series ring $R[[y;d]]$. Several of our main results examine when the rings R, R^d, and $R[[y;d]]$ are prime or semiprime under the assumption that d is a locally nilpotent derivation.

1. INTRODUCTION

The goal of this paper is to contrast the structure of a noncommutative algebra R with that of the skew power series ring $R[[y;\sigma,d]]$. We begin with a preview of our main results and then will define the terms and objects that will appear throughout this paper.

When d is a σ-derivation, much work has been done comparing the structure of R with the ring of invariants R^d and the skew polynomial ring $R[y;\sigma,d]$. Although there are some results in [1], relatively little has been done contrasting R and $R[[y;\sigma,d]]$. When examining a nonzero element of $R[y;\sigma,d]$, one can always look at its leading coefficient. This has proven to be a useful tool for comparing the structure of R with that of $R[y;\sigma,d]$. However, the sums in $R[[y;\sigma,d]]$ are infinite, therefore most elements do not have a leading coefficient. Thus new tools and techniques are needed to understand $R[[y;\sigma,d]]$. The primary tools we will use in this paper are to examine the action of $R[[y;\sigma,d]]$ on R and also to look at the trailing coefficients of elements of $R[[y;\sigma,d]]$.

In Section 2, we consider the case where d is a locally nilpotent, surjective q-skew σ-derivation. Theorem 1 shows that, regardless of the characteristic, R is a free right R^d-module of countably infinite rank and $R[[y;\sigma,d]]$ is isomorphic to $\mathrm{End}(R_{R^d})$. This result indicates that the relationship between $R[[y;\sigma,d]]$

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and R^d is often stronger than the one between $R[[y;\sigma,d]]$ and R. It follows, in Corollary 2, that $R[[y;\sigma,d]]$ is prime or semiprime if and only if R^d has these properties.

We also construct an example, in characteristic 0, of a prime algebra R with a locally nilpotent, surjective derivation d such that $R[[y;d]]$ is not semiprime. In addition, we show that examples exist where R^d is commutative. However, if we assume that nil subrings of R are nilpotent, then the primeness or semiprimeness of R and $R[[y;d]]$ are equivalent.

In Section 3, we drop the assumption that d is surjective and deal primarily with ordinary derivations. We first show that one direction of Corollary 2 no longer holds as we provide an example where R^d is not semiprime and $R[[y;d]]$ is prime. The main results of this section are Theorems 10 and 12 in which we show that if R^d is prime or semiprime, then so is $R[[y;d]]$.

We can now introduce the terminology that will be used throughout this paper. Let R be an algebra over a field K. If σ is a K-linear automorphism of R, then a σ-derivation d is a K-linear map $d : R \to R$ such that

$$d(rs) = d(r)s + \sigma(r)d(s),$$

for all $r, s \in R$.

There are two K-algebras closely related to R and d which have received a great deal of study. One is the ring of invariants R^d, which is defined as

$$R^d = \{r \in R \mid d(r) = 0\}.$$

Observe that R is a right module over R^d. The other algebra is the skew polynomial ring $R[y;\sigma,d]$, which consists of all formal sums

$$a_0 + a_1y + a_2y^2 + \cdots + a_ny^n,$$

where $n \geq 0$ and each $a_i \in R$. The algebra $R[y;\sigma,d]$ inherits all the relations of R along with the additional relation

$$yr = \sigma(r)y + d(r),$$

for all $r \in R$.

Observe that R is a left $R[y;\sigma,d]$-module. If $w = a_0+a_1y+a_2y^2+\cdots+a_ny^n \in R[y;\sigma,d]$ and $r \in R$, then we can define the action of w on r as

$$w(r) = a_0r + a_1d(r) + a_2d^2(r) + \cdots + a_nd^n(r).$$

We can now define the skew power series ring $R[[y;\sigma,d]]$ as all formal sums

$$a_0 + a_1y + a_2y^2 + \cdots ,$$

where each $a_i \in R$. Similar to the situation for $R[y;\sigma,d]$, the algebra $R[[y;\sigma,d]]$ inherits all the relations of R along with $yr = \sigma(r)y + d(r)$, for all $r \in R$. However, at this point, multiplication in $R[[y;\sigma,d]]$ is not necessarily well defined. To see this, let us look at the product $(1 + y + y^2 + \cdots)(r)$ in

$R[[y;\sigma,d]]$. Observe that when will pull r past the various powers of y, the constant term becomes the sum

$$r + d(r) + d^2(r) + \cdots$$

In general, this sum is not defined in R. Therefore, when discussing $R[[y;\sigma,d]]$, we will only consider σ-derivations which are locally nilpotent. This means that, for every $r \in R$, there exists $n = n(r) \geq 1$ such that $d^n(r) = 0$. This allows us to compute the constant term of $(1 + y + y^2 + \cdots)(r)$ as the sum only involves a finite number of nonzero terms.

If $\sigma = 1$ and d is an ordinary derivation, then d being locally nilpotent is sufficient to make multiplication in $R[[y;\sigma,d]]$ well defined. However, even if d is locally nilpotent, when we drop the assumption that $\sigma = 1$, another problem can arise. Let us again examine the product $(1 + y + y^2 + \cdots)(r)$ and we will now try to compute the coefficient of y. To this end, for every $n \geq 0$, let

$$p_n(r) = d^n\sigma(r) + d^{n-1}\sigma d(r) + \cdots + d\sigma d^{n-1}(r) + \sigma d^n(r).$$

Then the coefficient of y is

$$p_0(r) + p_1(r) + p_2(r) + \cdots$$

Note that even if d is locally nilpotent, this sum might not be defined in R.

If q is a nonzero element of K, we say that our σ-derivation is q-skew if

$$d\sigma(r) = q\sigma d(r),$$

for all $r \in R$. For any $n \geq 0$, let ker d^n denote the kernel of d^n. Observe that if d is q-skew, then σ is a bijection of ker d^n. It is easy to see, in this case, that the sum $p_0(r)+p_1(r)+p_2(r)+\cdots$ now contains only a finite number of nonzero terms. In fact, if d is locally nilpotent and q-skew then whenever we multiply elements in $R[[y;\sigma,d]]$, all the sums which arise when computing coefficients of powers of y only involve a finite number of nonzero terms. Therefore, when we examine $R[[y;\sigma,d]]$, we will always be assuming that d is a locally nilpotent q-skew σ-derivation. In this situation R also becomes a left $R[[y;\sigma,d]]$-module, for if $w = a_0 + a_1y + a_2y^2 + \cdots \in R[[y;\sigma,d]]$ and $r \in R$, we can define the action of w on r as

$$w(r) = a_0r + a_1d(r) + a_2d^2(r) + \cdots$$

If we let $\mathrm{End}(R_{R^d})$ denote the R^d-linear maps from R to R, then the action of $R[[y;\sigma,d]]$ on R defines a ring homomorphism

$$\psi : R[[y;\sigma,d]] \to \mathrm{End}(R_{R^d}).$$

2. Skew Derivations - The Surjective Case

In this section, we look at the relationship between R, R^d, and $R[[y;\sigma,d]]$ in the important special case where our locally nilpotent q-skew σ-derivation

d is surjective. In Theorems 10 and 12 of the next section, we will see that the assumption that d is surjective is superfluous under certain conditions.

We begin with a result which indicates a very tight connection between R, R^d, and $R[[y;\sigma,d]]$.

Theorem 1. *Let R be an algebra with a q-skew σ-derivation d which is locally nilpotent and surjective. Then R is a free right module of countably infinite rank over the invariant subring R^d and the skew power series ring $R[[y;\sigma,d]]$ is isomorphic to the endomorphism ring* $\mathrm{End}(R_{R^d})$.

Proof. Let $x_0 = 1$; then the surjectivity of d allows us to construct a sequence $x_0, x_1, x_2, \ldots$ such that $d(x_i) = x_{i-1}$, for all $i \geq 1$. We will now verify, by induction, that the kernel of d^n is equal to the direct sum

$$x_0R^d \oplus x_1R^d \oplus \cdots \oplus x_{n-1}R^d,$$

for every $n \geq 1$.

The $n = 1$ case is clear, therefore we may assume the result holds for some $n \geq 1$ and we must show that the kernel of d^{n+1} is

$$x_0R^d \oplus x_1R^d \oplus \cdots \oplus x_{n-1}R^d \oplus x_nR^d.$$

To show that this sum is direct, we need to show that

$$x_nR^d \cap (x_0R^d \oplus x_1R^d \oplus \cdots \oplus x_{n-1}R^d) = 0.$$

However, if $r \in R^d$ such that

$$x_nr \in x_0R^d \oplus x_1R^d \oplus \cdots \oplus x_{n-1}R^d = \ker\, d^n,$$

we have

$$0 = d^n(x_nr) = d^n(x_n)r = x_0r = r.$$

Thus $r = 0$, hence $x_nr = 0$, and the intersection above is indeed equal to 0.

Next, since $d^{n+1}(x_n) = 0$, it follows that

$$d^{n+1}(x_nR^d) = d^{n+1}(x_n)R^d = 0.$$

Thus $x_nR^d \subset \ker\, d^{n+1}$, hence

$$x_0R^d \oplus x_1R^d \oplus \cdots \oplus x_{n-1}R^d \oplus x_nR^d \subset \ker\, d^n + \ker\, d^{n+1} \subset \ker\, d^{n+1}.$$

For the reverse inclusion, suppose $a \in \ker\, d^{n+1}$. Therefore $d(a) \in \ker\, d^n$, hence

$$d(a) = x_0r_1 + x_1r_2 + \cdots + x_{n-1}r_n,$$

where each $r_i \in R^d$. If we let

$$r_0 = a - (x_1r_1 + \cdots + x_nr_n), \tag{1}$$

then

$$\begin{aligned} d(r_0) &= d(a - (x_1 r_1 + \cdots + x_n r_n)) \\ &= d(a) - (d(x_1) r_1 + \cdots + d(x_n) r_n) \\ &= d(a) - (x_0 r_1 + \cdots + x_{n-1} r_n) = 0. \end{aligned}$$

Therefore $r_0 \in R^d$ and it now follows from (1) that

$$a = x_0 r_0 + x_1 r_1 + \cdots + x_n r_n \in x_0 R^d \oplus x_1 R^d \oplus \cdots \oplus x_{n-1} R^d \oplus x_n R^d.$$

Thus the kernel of d^n is equal to $x_0 R^d \oplus x_1 R^d \oplus \cdots \oplus x_{n-1} R^d$.

Since d is locally nilpotent, $R = \bigcup_{n=1}^{\infty} \ker\, d^n$. In light of the previous argument, we know now that

$$R = \oplus_{i=0}^{\infty} x_i R^d.$$

Thus R is a free right module of countably infinite rank over R^d. Furthermore, the action of every element of $R[[y; \sigma, d]]$ and $\mathrm{End}(R_{R^d})$ on R is completely determined by its action on the sequence $x_0, x_1, x_2, \ldots$. It now suffices to show that the homomorphism

$$\psi : R[[y; \sigma, d]] \to \mathrm{End}(R_{R^d})$$

induced by the action of $R[[y; \sigma, d]]$ on R is both injective and surjective.

If the power series

$$f = a_0 + a_1 y + a_2 y^2 + \cdots$$

acts as 0 on R then, for all $n \geq 0$, we have

$$\begin{aligned} 0 &= f(x_n) = (a_0 + a_1 y + a_2 y^2 + \cdots)(x_n) \qquad (2) \\ &= a_0 x_n + a_1 x_{n-1} + \cdots + a_{n-1} x_1 + a_n x_0. \end{aligned}$$

Letting $n = 0$ immediately tells us that $a_0 = 0$. Furthermore, if we already know that $0 = a_0 = a_1 = \cdots = a_{n-1}$, then (2) tells us that $a_n = 0$. Thus induction asserts that $f = 0$ and so, the homomorphism ψ is injective.

Finally, suppose $w \in \mathrm{End}(R_{R^d})$; we need to construct some $f \in R[[y; \sigma, d]]$ whose action on R is the same as that of w. To this end, for $n \geq 0$, let $t_n = w(x_n)$ and now construct a sequence of elements of R as follows:

$$a_0 = t_0$$

and if $a_0, a_1, \ldots, a_n$ have already been constructed, let

$$a_{n+1} = t_{n+1} - (a_0 x_{n+1} + a_1 x_n + \cdots + a_n x_1).$$

Using the above sequence, we can let f be the power series $a_0 + a_1 y + a_2 y^2 + \cdots$.

Recall that $d^n(x_m) = x_{m-n}$, for all $n \leq m$, and $d^n(x_m) = 0$, whenever $n > m$. Combining these facts with the construction of the sequence $a_0, a_1, a_2 \ldots$, we have

$$f(x_0) = a_0 x_0 = a_0 = t_0 = w(x_0)$$

and

$$f(x_{n+1}) = a_0x_{n+1} + a_1x_n + \cdots + a_nx_1 + a_{n+1}x_0 = t_{n+1} = w(x_{n+1}).$$

Thus the action on R of the series f is the same as that of w, hence ψ is also surjective. Thus $R[[y;\sigma,d]] \approx \operatorname{End}(R_{R^d})$, as desired.

□

Theorem 1 indicates that there is a particularly close relationship between the structure of R^d and that of $R[[y;\sigma,d]]$. In the prime and semiprime cases, we record this as

Corollary 2. *Let R be an algebra with a q-skew σ-derivation d which is locally nilpotent and surjective and let R^d denote the kernel of d.*

(1) *The skew power series ring $R[[y;\sigma,d]]$ is prime if and only if R^d is prime.*
(2) *The skew power series ring $R[[y;\sigma,d]]$ is semiprime if and only if R^d is semiprime.*

Proof. By Theorem 1, R is a free right module of countably infinite rank over R^d and $R[[y;\sigma,d]]$ is isomorphic to the endomorphism ring $\operatorname{End}(R_{R^d})$. Therefore, $R[[y;\sigma,d]]$ can be viewed as the ring of countably infinite matrices over R^d where each column has only a finite number of nonzero entries. This implies that $R[[y;\sigma,d]]$ is prime or semiprime if and only if R^d has the same property. □

The relationship between a ring R with a skew derivation d and the skew polynomial ring $R[y;\sigma,d]$ has been extensively studied for many years. In particular, it is well known that if R is prime or semiprime, then $R[y;\sigma,d]$ inherits these properties. It is somewhat surprising that for skew power series rings, the relationship between R^d and $R[[y;\sigma,d]]$ appears to be stronger than the relationship between R and $R[[y;\sigma,d]]$. In light of Corollary 2 and the known results on skew polynomial rings, one might suspect that if R is prime, then $R[[y;\sigma,d]]$ would also be prime. However, the next example shows that even for ordinary surjective derivations in characteristic 0, it is possible for R to be prime and for $R[[y;d]]$ to fail to be semiprime. In fact, an example exists in which R^d is commutative.

Example 3. *A prime algebra R of characteristic 0 with a locally nilpotent, surjective derivation d such that $R[[y;d]]$ is not semiprime. In addition, R can be chosen such that R^d is commutative.*

Proof. Let K be a field of characteristic 0 and let B be the Grassmann algebra over K generated by the countably infinite set $e_1, e_2, \ldots$. Recall that $e_ie_j = -e_je_i$, for all i,j. The K-linear function δ defined as $\delta(e_i) = e_{i+1}$, for all $i \geq 1$, extends to a derivation of B. It was shown in [3] that although B is not semiprime, the skew polynomial ring $B[x;\delta]$ is prime.

We can now let $R = B[x;\delta]$ and can define the derivation d of R as $d(B) = 0$ and $d(x) = 1$. It is not difficult to see that d is a locally nilpotent, surjective derivation of R with $B = R^d$. Since R^d is not semiprime, Corollary 2 asserts that although R is prime, $R[[y;d]]$ is not semiprime.

If we let A denote the center of B, then δ restricts to A. It is also shown in [3] that $A[x;\delta]$ is prime even though A is commutative and not semiprime. If we had instead let $R = A[x;\delta]$, then the identical reasoning as above tells us that the function d defined as $d(A) = 0$ and $d(x) = 1$ is a locally nilpotent, surjective derivation of the prime ring R such that R^d is commutative and $R[[y;d]]$ is not semiprime.

□

In various types of rings, such as rings with the ascending chain condition on left and right annihilators or one-sided Goldie rings, nil subrings are nilpotent [4], [5]. The next result shows that if nil subrings of R are nilpotent, then the properties of being prime and semiprime are indeed inherited by $R[[y;d]]$ from R.

Theorem 4. *Let R be an algebra of characteristic* 0 *with a locally nilpotent, surjective derivation d.*

(1) *if $R[[y;d]]$ is (semi)prime then R is (semi)prime*
(2) *if nil subrings of R are nilpotent and if R is (semi)prime then $R[[y;d]]$ is (semi)prime.*

Proof. In order to prove (1), let us first assume that $R[[y;d]]$ is prime or semiprime. Then Corollary 2 asserts that R^d is also prime or semiprime. However, we can now apply Lemma 2.1 in [2] to see that $R = R^d[x;\delta]$, for some derivation δ of R^d. However, if R^d is prime or semiprime, then it is well known that the skew polynomial ring $R^d[x;\delta]$ is also prime or semiprime. Hence R is prime or semiprime, as desired.

For (2), as in the previous paragraph, $R = R^d[x;\delta]$, where δ is a derivation of R^d. Since R is a skew polynomial ring over R^d, if R is semiprime then R^d is δ-semiprime. This means that R^d has no nonzero nilpotent δ-stable ideals. Now, let N denote the sum of all the nil ideals of R^d. Since R has characteristic 0, N is a δ-stable ideal of R^d. However, since N is a nil subring of R, it must be nilpotent. The fact that R^d is δ-semiprime immediately implies that $N = 0$, hence R^d is semiprime. By Corollary 2, it now follows that $R[[y;d]]$ is semiprime.

Finally, suppose R is prime. By the argument in the previous paragraph, R^d is semiprime. In addition, since $R = R^d[x;\delta]$, the primeness of R implies that R^d is δ-prime. Thus the annihilator of every nonzero δ-stable ideal of R^d is zero. If $J \neq 0$ is an ideal of R^d, let $I = \{a \in R^d \mid aJ = 0\}$. Since

$$I\delta(J^2) \subseteq I(\delta(J)J + J\delta(J)) \subseteq IJ = 0,$$

we have

$$(\delta(I)J)^2 \subseteq \delta(I)J^2 = \delta(IJ^2) = 0.$$

The semiprimeness of R^d now implies that $\delta(I)J = 0$. Thus $\delta(I) \subseteq I$, hence I is a δ-stable ideal of R^d with a nonzero annihilator. As a result, $I = 0$, which tells us that the annihilator of every nonzero ideal of R^d is 0. Hence R^d is prime. It now follows, by Corollary 2, that $R[[y; d]]$ is also prime.

□

As another application of Theorem 1, we show that nonisomorphic algebras with nonisomorphic skew polynomial rings can still result in isomorphic skew power series rings.

Example 5. *Noetherian domains R_1 and R_2 of characteristic 0 with locally nilpotent, surjective derivations d_1, d_2, respectively, such that R_1 and R_2 are not isomorphic, $R_1[y; d_1]$ and $R_2[y; d_2]$ are not isomorphic, yet $R_1[[y; d_1]]$ and $R_2[[y; d_2]]$ are isomorphic.*

Proof. Let K be a field of characteristic 0 and let R_1 be the commutative polynomial ring over K generated by s_1, t_1. Then let R_2 be the K-algebra generated by s_2, t_2 with the relation $s_2t_2 - t_2s_2 = s_2$. Thus R_2 is the enveloping algebra of the 2-dimensional nonabelian Lie algebra. Since R_2 is not commutative, R_1 and R_2 are clearly not isomorphic.

For $i = 1, 2$, let d_i be the K-linear function defined as $d_i(s_i) = 0$ and $d_i(t_i) = 1$. It is not hard to check that for both values of i, d_i extends to a locally nilpotent, surjective derivation of R_i with $(R_i)^{d_i} = K[s_i]$. Thus $(R_1)^{d_1}$ and $(R_2)^{d_2}$ are isomorphic. Theorem 1 now implies that $R_1[[y; d_1]]$ and $R_2[[y; d_2]]$ are isomorphic.

It now suffices to show that $R_1[y; d_1]$ and $R_2[y; d_2]$ are not isomorphic. We will do this by comparing their centers. To this end, if the center of $R_2[y; d_2]$ is not contained in R_2, let w be an element in the center of $R_2[y; d_2]$ which has the smallest degree in y of those not in R_2. Therefore, we can write

$$w = a_0 + a_1y + \cdots + a_{n-1}y^{n-1} + a_ny^n,$$

where each $a_i \in R_2$ and $n \geq 1$. If $r \in R_2$, then commuting w with r yields

$$0 = [r, w] = b_0 + b_1y + \cdots + b_{n-1}y^{n-1} + [r, a_n]y^n,$$

where each $b_i \in R_2$. Therefore $[r, a_n] = 0$, hence a_n is central in R_2. It is well known that the center of the R_2 is the field K, therefore without loss of generality we may assume that $a_n = 1$.

If $r \in R_2$, then since $yr = ry + d_2(r)$, it follows that

$$y^nr = ry^n + nd_2(r)y^{n-1} + c_{n-2}y^{n-2} + \cdots + c_1y + c_0,$$

where each $c_i \in R_2$. Therefore, commuting w with r now yields

$$\begin{aligned} 0 = [w, r] &= [a_0, r] + a_1[y, r] + [a_1, r]y \\ &\quad + \cdots + a_{n-1}[y^{n-1}, r] + [a_{n-1}, r]y^{n-1} + [y^n, r] \\ &= f_0 + f_1 y + \cdots + f_{n-2}y^{n-2} + ([a_{n-1}, r] + nd_2(r))y^{n-1}. \end{aligned}$$

The previous equation tells us that

$$d_2(r) = r(\frac{a_{n-1}}{n}) - (\frac{a_{n-1}}{n})r,$$

for all $r \in R_2$. As a result, the derivation d_2 is inner, However, this leads to a contradiction as it is easy to check that there is no element of R_2 which can be commuted with t_2 and give an answer of 1. Hence, it is indeed the case that the center of $R_2[y; d_2]$ is contained in R_2. But since the center of R_2 is K, we now know that the center of $R_2[y; d_2]$ is K.

On the other hand, $K[s_1]$ is central and R_1 and $d_1(K[s_1]) = 0$, thus $K[s_1]$ is central in $R_1[y; d_1]$. Since the centers of $R_1[y; d_1]$ and $R_2[y; d_2]$ are not isomorphic, it follows that $R_1[y; d_1]$ and $R_2[y; d_2]$ are not isomorphic.

□

As we look back at Example 5, we note that since $K[s_1]$ is central in $R_1[y; d_1]$, it is also central in $R_1[[y; d_1]]$. Therefore, despite the fact that the center of $R_2[y; d_2]$ is K, the center of $R_2[[y; d_2]]$ contains a polynomial ring. Since none of the nonconstant polynomials in $R_2[y; d_2]$ are central, it raises the question as to what do some of the central elements of $R_2[[y; d_2]]$ look like?

Recall that $R_2[[y; d_2]]$ has three relations

$$[s_2, t_2] = s_2, \quad [y, s_2] = 0, \quad [y, t_2] = 1.$$

The third relation says that when a power series with coefficients in K is commuted with t_2, the answer is the derivative of the series with respect to y. Therefore, if we let e^{-y} denote the familiar power series from calculus, we have

$$[e^{-y}, t_2] = -e^{-y}.$$

This implies that

$$[s_2 e^{-y}, t_2] = s_2[e^{-y}, t_2] + [s_2, t_2]e^{-y} = s_2(-e^{-y}) + s_2(e^{-y}) = 0.$$

Therefore $s_2 e^{-y}$ commutes with t_2. Since s_2 and y commute, it follows that the series

$$s_2 e^{-y} = s_2 - s_2 y + \frac{1}{2}s_2 y^2 - \frac{1}{6}s_2 y^3 + \cdots$$

is central in $R_2[[y; d_2]]$. Therefore, for every $n \in \mathbb{N}$, the series $(s_2 e^{-y})^n = (s_2)^n e^{-ny}$ is also central in $R_2[[y; d_2]]$.

Theorem 1 showed us that if d is locally nilpotent and surjective, then the skew power series ring cannot be a domain. In the next example, we will see that it is still quite common for locally nilpotent derivations to produce skew power series rings which are domains.

Theorem 6. *Let K be a field of characteristic 0 and let d be a locally nilpotent derivation of the commutative polynomial ring $R = K[t][x]$ such that $d(K[t]) = 0$ and $d(x) \in K[t]$. Then the skew power series ring $R[[y; d]]$ is a domain if and only if $d(x)$ is not a nonzero element of K.*

Proof. In one direction, if $d(x) = \alpha \in K^*$, then when we let $x_1 = x\alpha^{-1}$, we have that $R = K[t][x_1]$ with $d(K[t]) = 0$ and $d(x_1) = 1$. By Lemma 2.1 in [2] d is surjective. Theorem 1 now asserts that $R[[y; d]]$ is the endomorphism ring of $K[t][x_1]$ as a module over $K[t]$ and so, $R[[y; d]]$ is not a domain.

In the other direction, we need to consider when $d(x)$ is either 0 or a polynomial of degree at least 1 in $K[t]$. If $d(x) = 0$, then $R[[y; d]]$ is an ordinary power series ring in the variable y with coefficients in the domain $K[t][x]$. Therefore, in this case, $R[[y; d]]$ is a domain. As a result, it remains to consider the case where $d(x) \in K[t]$ has degree at least one. We can let $p(t) \in K[t]$ be a irreducible polynomial which divides $d(x)$. Now suppose $f \in K[t][x]$; we can write

$$f = \sum_{i=0}^{l} f_i(t)x^i,$$

where each $f_i(t) \in K[t]$. Then there exists a largest integer $j \geq 0$ such that $p(t)^j$ divides each $f_i(t)$. In particular,

$$f = p(t)^j \sum_{i=0}^{l} g_i(t)x^i,$$

where at least one $g_i(t)$ is not divisible by $p(t)$.

Let u, v be nonzero elements of $R[[y; d]]$; we need to show that $uv \neq 0$. We can write

$$u = \sum_{i=0}^{\infty} a_i y^i \quad \text{and} \quad v = \sum_{i=0}^{\infty} b_i y^i,$$

where each $a_i, b_i \in K[t][x]$. Arguing as above, there exist largest integers $n, m \geq 0$ such that we can rewrite u and v as

$$u = p(t)^n \sum_{i=0}^{\infty} {a_i}^* y^i \quad \text{and} \quad v = p(t)^m \sum_{i=0}^{\infty} {b_i}^* y^i,$$

where ${a_i}^*, {b_i}^* \in K[t][x]$ and at least one ${a_i}^*$ and at least one ${b_j}^*$ cannot have an additional $p(t)$ factored out. Observe that $p(t)$ is central and not a zero divisor in $R[[y; d]]$. Therefore, it we let

$$u^* = \sum_{i=0}^{\infty} {a_i}^* y^i \quad \text{and} \quad v^* = \sum_{i=0}^{\infty} {b_i}^* y^i,$$

then $uv = 0$ if and only if $u^*v^* = 0$.

Next, let I be the ideal of $R[[y;d]]$ generated by $p(t)$. Since $[y,x] = d(x) \in I$, the image of y is central in the quotient ring $(R[[y;d]])/I$. If we let L denote the field $K[t]/(p(t))$, then it is easy to see that $(R[[y;d]])/I$ is isomorphic to $L[x][[y]]$. Thus $(R[[y;d]])/I$ is a domain as it is an ordinary power series over the commutative domain $L[x]$. Since at least one coefficient of both u^* and v^* are not divisible by $p(t)$, their images in $(R[[y;d]])/I$ are both nonzero. Therefore

$$(u^* + I)(v^* + I) \neq 0$$

in $(R[[y;d]])/I$. This tells us that the product u^*v^* is certainly nonzero in $R[[y;d]]$. Hence $uv \neq 0$ in $R[[y;d]]$ and so, $R[[y;d]]$ is a domain.

□

The next example will illustrate some of the differences between characteristic p and characteristic 0 for ordinary derivations. In the characteristic 0 case, it was shown in the first part of Theorem 4 that if d is a locally nilpotent, surjective derivation such that $R[[y;d]]$ was prime, then R needed to be prime. However, as we shall soon see, this is certainly not the case in characteristic $p > 0$ as we exhibit an algebra R which is not semiprime, yet $R[[y;d]]$ is primitive.

Example 7. *A commutative algebra R in characteristic $p > 0$ with a nil ideal of codimension 1 and a locally nilpotent, surjective derivation d such that R^d is a field, $R[y;d]$ is simple, and $R[[y;d]]$ is primitive but not simple.*

Proof. Let K be a field of characteristic $p > 0$ and let R be the K-algebra generated by the commuting variables $x_0, x_1, x_2, \dots$ such that ${x_i}^p = 0$, for all $i \geq 0$. It is clear that R is commutative with a nil ideal of codimension 1. Next, define the derivation d as $d(x_0) = 1$ and $d(x_{i+1}) = {x_0}^{p-1}{x_1}^{p-1}\cdots{x_i}^{p-1}$, for all $i \geq 0$.

It then follows that if $j_0 \geq 1$, then

$$d({x_0}^{j_0}{x_1}^{j_1}\cdots{x_m}^{j_m}) = j_0{x_0}^{j_0-1}{x_1}^{j_1}\cdots{x_m}^{j_m},$$

and if $0 < i_1 < \cdots < i_m$ with $j_1 \geq 1$, then

$$d({x_{i_1}}^{j_1}{x_{i_2}}^{j_2}\cdots{x_{i_m}}^{j_m}) = j_1{x_0}^{p-1}{x_1}^{p-1}\cdots{x_{i_1-1}}^{p-1}{x_{i_1}}^{j_1-1}{x_{i_2}}^{j_2}\cdots{x_{i_m}}^{j_m}.$$

Another way to look at this is, if $n \in \mathbb{N}$, let

$$n = i_0 + i_1 \cdot p + i_2 \cdot p^2 + \cdots + i_m \cdot p^m,$$

be the p-adic expansion of n and let Δ_n be the monomial

$$\Delta_n = {x_0}^{i_0}{x_1}^{i_1}\cdots{x_m}^{i_m}.$$

Then $d(\Delta_n) = \beta\Delta_{n-1}$, where β is a nonzero element of K.

It is now easy to see that d is locally nilpotent, d is surjective, $R^d = K$, and R is d-simple. Therefore R^d is a field and $R[y;d]$ is simple. Furthermore, Theorem 1 asserts that $R[[y;d]]$ is the entire ring of K-linear transformations of R, hence $R[[y;d]]$ is primitive.

□

3. Derivations - The General Case

In Section 2, we examined the close relationship between the structure of R^d and that of $R[[y;\sigma,d]]$ when d is a locally nilpotent, surjective q-skew σ-derivation. In particular, Corollary 2 asserted that $R[[y;\sigma,d]]$ is prime or semiprime if and only if R^d has the same properties. It is now natural to wonder if this remains true if we no longer assume that d is surjective. Our next example shows that if d is an ordinary locally nilpotent derivation which is not surjective, then it is possible for $R[[y;d]]$ to be prime even when R^d is not semiprime.

Example 8. *A prime algebra R of arbitrary characteristic with a derivation d such that $d^3=0$, $R[y;d]$ and $R[[y;d]]$ are both prime, yet R^d is not semiprime.*

Proof. Let R be any prime algebra which is not a domain and let a be a nonzero element of R such that $a^2=0$. If d is the inner derivation of R induced by a, then $d^3=0$. Furthermore, since a is central in R^d, it follows that R^d is not semiprime.

In both $R[y;d]$ and $R[[y;d]]$, we can let $Y=y-a$. Then Y is central in both $R[y;d]$ and $R[[y;d]]$, furthermore

$$y^n=((y-a)+a)^n=(y-a)^n+na(y-a)^{n-1}=Y^n+naY^{n-1}.$$

Therefore the series $\sum_{i=0}^{\infty} b_i y^i$ can be rewritten as

$$\sum_{i=0}^{\infty}(b_i+(i+1)a)Y^n.$$

As a result, $R[y;d]=R[Y]$ and $R[[y;d]]=R[[Y]]$. Since $R[Y]$ and $R[[y]]$ are an ordinary polynomial ring and an ordinary power series over the prime ring R, it is immediate that $R[y;d]$ and $R[[y;d]]$ are both prime.

□

For most of this section, we will restrict our attention to ordinary derivations. In the main result of this section, we will show that one direction of Corollary 2 still holds for locally nilpotent derivations, even if d is not surjective. In particular, we will show that if R^d is prime or semiprime, then so is $R[[y;d]]$. In showing this, we will need to pay close attention to the case where d is nilpotent, which is clearly a situation which does not arise when d is surjective. We begin with

Lemma 9. *Let R be an algebra with a locally nilpotent derivation d and, for every $t\geq 0$, let $A_t=\{d^t(r)\mid r\in\ker d^{t+1}\}$.*

(1) *If* $d^t(R) \neq 0$*, then* A_t *is a nonzero ideal of* R^d*.*
(2) *If* R *has characteristic* 0*,* R^d *is semiprime, and* $d^n(L) = 0$ *where* $L \neq 0$ *is a* d*-stable left ideal of* R*, then* $d(LR) = 0$*.*
(3) *If* R *has characteristic* $p > 0$*,* R^d *is semiprime, and* $d^n(L) = 0$ *where* $L \neq 0$ *is a* d*-stable left ideal of* R*, then the index of nilpotence of* d *on both* L *and* LR *is* p^l*, for some* $l \geq 0$*.*

Proof. Since d is locally nilpotent, if $d^t \neq 0$ there exists $r \in R$ such that $d^{t+1}(r) = 0$ and $0 \neq d^t(r) \in A_t$. Thus A_t is nonzero. It is easy to see that if $r \in \ker\ d^{t+1}$ and $a \in R^d$, then $ar, ra \in \ker\ d^{t+1}$. Furthermore,

$$ad^t(r) = d^t(ar) \quad \text{and} \quad d^t(r)a = d^t(ra).$$

Thus $ad^t(r), d^t(r)a \in A_t$ and so, A_t is an ideal of R^d, thereby proving part (1).

For part (2), we may assume that n is the index of nilpotence of d on L. If $n > 1$, we have

$$0 = d^n(Ld^{n-2}(L)) = nd^{n-1}(L)d^{n-1}(L).$$

However, in this situation, $d^{n-1}(L)$ is a nonzero left ideal of the semiprime algebra R^d in characteristic 0, which leads to the contradiction $nd^{n-1}(L) \cdot d^{n-1}(L) \neq 0$. As a result, $n = 1$ which implies that

$$0 = d(RL) = d(R)L.$$

However, this tells us that

$$(Ld(R))^2 = Ld(R)Ld(R) = 0.$$

Therefore $Ld(R)$ is a d-stable ideal of R of square 0. We claim that $Ld(R) = 0$. If this is not the case, then since d is locally nilpotent, it follows that $Ld(R) \cap R^d \neq 0$. But $Ld(R) \cap R^d$ is a nonzero left ideal of square zero in the semiprime ring R^d, a contradiction. Thus $Ld(R) = 0$ and we now have

$$d(LR) = d(L)R + Ld(R) = 0,$$

proving (2).

For part (3), we may assume that n is the index of nilpotence of d and we can write $n = p^l m$, where p does not divide m. We will first show that $m = 1$. By way of contradiction, if $m > 1$, we can let $\delta = d^{p^l}$. Then δ is a derivation such that $\delta^m(L) = 0$. This implies that

$$0 = \delta^m(L\delta^{m-2}(L)) = m\delta^{m-1}(L)\delta^{m-1}(L).$$

Observe that $p^l(m-1) \leq p^l m - 1 = n - 1$. Combining this with that fact that L is d-stable, we have $d^{n-1}(L) \subseteq \delta^{m-1}(L)$. This tells us that

$$md^{n-1}(L)d^{n-1}(L) = 0.$$

However this is a contradiction as $d^{n-1}(L)$ is a nonzero left ideal of the semiprime algebra R^d and the characteristic does not divide m. Thus $n = p^l$.

The remainder of the proof is very similar to the argument in part (2). Observe that since $n = p^l$, d^n is a derivation which vanishes on L. Arguing as above, $Ld^n(R)$ must be zero, otherwise $Ld^n(R) \cap R^d$ would be a nonzero left ideal of square zero in the semiprime ring R^d. Therefore

$$d^{p^l}(LR) = d^n(LR) = d^n(L)R + Ld^n(R) = 0,$$

as required.

□

We can now handle the case where R^d is prime. Observe that if d is locally nilpotent but not nilpotent then it remains true, as in the surjective case, that the action of $R[[y;d]]$ on R is faithful.

Theorem 10. *Let R be an algebra with a locally nilpotent derivation d such that R^d is prime. Then the skew power series ring $R[[y;d]]$ is prime. In addition, if d is not nilpotent then the action of $R[[y;d]]$ on R is faithful.*

Proof. We first consider the case where d is not nilpotent. If $J \neq 0$ is an ideal of $R[[y;d]]$, let

$$w = a_t y^t + a_{t+1} y^{t+1} + \cdots$$

be an element of J, where each $a_i \in R$ and $a_t \neq 0$. Observe that

$$[y, w] = d(a_t)y^t + d(a_{t+1})y^{t+1} + \cdots$$

and

$$[y, [y, w]] = d^2(a_t)y^t + d^2(a_{t+1})y^{t+1} + \cdots$$

are also elements of J. Therefore, by starting with w and continuing to bracket with y, we can produce an element of J where the coefficient of y^t is a nonzero element of R^d. Hence, without loss of generality, we may assume that w has been chosen so that $0 \neq a_t \in R^d$.

If we let w act on elements of ker d^{t+1}, we obtain

$$w(\ker\ d^{t+1}) = a_t A_t.$$

By Lemma 9(1), A_t is a nonzero ideal of the prime ring R^d. Thus $a_t A_t \neq 0$. In particular, w does not vanish on all of R. As a result, given any nonzero ideal of $R[[y;d]]$, we can produce an element in the ideal which does not vanish on R. Hence the action of $R[[y;d]]$ is faithful on R.

Now suppose I, J are nonzero ideals of $R[[y;d]]$. Arguing as above, we may assume that there exist $v \in I$, $w \in J$ such that

$$v = b_s y^s + b_{s+1} y^{s+1} + \cdots$$

and

$$w = a_t y^t + a_{t+1} y^{t+1} + \cdots ,$$

where the coefficients of v and w belong to R and b_s, a_t are nonzero elements of R^d. If we let $r_1 \in \ker\ d^{s+1}$ and $r_2 \in \ker\ d^{t+1}$, then $vr_1w \in IJ$. When we let this element act on r_2, we obtain

$$\begin{aligned} vr_1w(r_2) &= vr_1(w(r_2)) = vr_1(a_t d^t(r_2)) = v(r_1 a_t d^t(r_2)) \\ &= b_s d^s(r_1 a_t d^t(r_2)) = b_s d^s(r_1) a_t d^t(r_2) \\ &\in b_s A_s a_t A_t \neq 0. \end{aligned}$$

Since $b_s A_s a_t A_t \neq 0$, we can choose r_1, r_2 such that $vr_1w(r_2) \neq 0$. But this tells us that $vr_1w \neq 0$, hence $IJ \neq 0$. Thus $R[[y; d]]$ is prime.

It remains to consider the case where d is nilpotent. If R has characteristic 0, then Lemma 9(2) asserts that $d = 0$. Therefore $R = R^d$ and $R[[y; d]]$ is an ordinary power series over a prime ring. Hence $R[[y; d]]$ is prime. Therefore, we may now assume that R has characteristic $p > 0$. Lemma 9(3) now tells us that the index of nilpotence is p^l, where $l \geq 0$. Let I, J be nonzero ideals of $R[[y; d]]$. We may once again assume that there exist $v \in I$, $w \in J$ such that

$$v = b_s y^s + b_{s+1} y^{s+1} + \cdots$$

and

$$w = a_t y^t + a_{t+1} y^{t+1} + \cdots , \tag{3}$$

where the coefficients of v and w belong to R and b_s, a_t are nonzero elements of R^d.

Since the characteristic is p and $d^{p^l} = 0$, the element y^{p^l} is both central and regular in $R[[y; d]]$. If s and t are the integers in (3), we can apply the division algorithm and divide each of them by p^l to obtain

$$s = q_1 p^l + s_1 \quad \text{and} \quad t = q_2 p^l + s_2,$$

where $0 \leq s_1, t_1 < p^l$. We can now factor $y^{q_1 p^l}$ out of v and $y^{q_2 p^l}$ out of w in (3) to obtain,

$$v = y^{q_1 p^l} v_1 \quad \text{and} \quad w = y^{q_2 p^l} w_1,$$

where

$$v_1 = b_s y^{s_1} + b_{s+1} y^{s_1+1} + \cdots$$

and

$$w_1 = a_t y^{t_1} + a_{t+1} y^{t_1+1} + \cdots .$$

Now let $r_1 \in \ker\ d^{s_1+1}$ and $r_2 \in \ker\ d^{t_1+1}$; if we let $v_1 r_1 w_1$ act on r_2, we obtain

$$\begin{aligned} v_1 r_1 w_1(r_2) &= v_1 r_1(w_1(r_2)) = v_1 r_1(a_t d^{t_1}(r_2)) = v_1(r_1 a_t d^{t_1}(r_2)) \\ &= b_s d^{s_1}(r_1 a_t d^{t_1}(r_2)) = b_s d^{s_1}(r_1) a_t d^{t_1}(r_2) \in b_s A_{s_1} a_t A_{t_1}. \end{aligned}$$

Since both s_1 and t_1 are less than p^l, Lemma 9(1) tells us that A_{s_1} and A_{t_1} are nonzero ideals of the prime ring R^d. Therefore, $b_s A_{s_1} a_t A_{t_1} \neq 0$, thus we can choose r_1, r_2 such that $v_1r_1w_1(r_2) \neq 0$. Combining the facts that $v_1r_1w_1 \neq 0$ and y^{p^l} is both central and regular, it follows that

$$vr_1w = (y^{q_1p^l}v_1)r_1(y^{q_2p^l}w_1) = y^{(q_1+q_2)p^l}(v_1r_1w_1) \neq 0.$$

Since $vr_1w \in IJ$, we see that $IJ \neq 0$. Hence $R[[y;d]]$ is prime, thereby concluding the proof.

□

In order to deal with semiprime rings, we also need

Lemma 11. *Let $w = a_ty^t + a_{t+1}y^{t+1} + \cdots \in R[[y;d]]$, where each $a_i \in R$ and a_t is a nonzero element of R^d. If R^d is semiprime and $a_tA_t \neq 0$, then $wRw \neq 0$. Therefore, if I is an ideal of $R[[y;d]]$ and $w \in I$, then $I^2 \neq 0$.*

Proof. Observe that a_tA_t is a nonzero right ideal of R^d. Since R^d is semiprime, we know that

$$a_tA_ta_tA_t \neq 0.$$

Therefore there exist $r_1, r_2 \in \ker\, d^{t+1}$ such that

$$a_td^t(r_1)a_td^t(r_2) \neq 0.$$

If we let the element $wr_1w \in R[[y;d]]$ act on $r_2 \in R$, then we obtain

$$wr_1w(r_2) = wr_1(a_td^t(r_2)) = w(r_1a_td^t(r_2)) = a_td^t(r_1)a_td^t(r_2) \neq 0.$$

Thus $wr_1w \neq 0$ and hence $wRw \neq 0$.

□

We can now prove the main result of this section.

Theorem 12. *Let R be an algebra with a locally nilpotent derivation d. If R^d is semiprime, then the skew power series ring $R[[y;d]]$ is semiprime.*

Proof. If $I \neq 0$ is an ideal of $R[[y;d]]$, let $w = a_ty^t + a_{t+1}y^{t+1} + \cdots + \in I$, where each $a_i \in R$ and a_t is a nonzero element of R^d. By Lemma 11, if $a_tA_t \neq 0$, then $I^2 \neq 0$. Therefore, it suffices to consider the case where $a_tA_t = 0$. In this situation, we can let $L = \{r \in R \mid rA_t = 0\}$ and observe that L is a d-stable left ideal of R which contains a_t.

If $d^t(L) \neq 0$ then the fact the L is d-stable implies that there exists $r \in L$ such that $d^t(r) \neq 0$ and $d^{t+1}(r) = 0$. Thus

$$0 \neq d^t(r) \in L \cap A_t.$$

Therefore $L \cap A_t$ is a nonzero left ideal of R^d such that

$$(L \cap A_t)^2 \subseteq LA_t = 0,$$

contrary to the fact that R^d is semiprime. As a result, we may now assume that $d^t(L) = 0$.

If we let $M = LR$, then since R^d is semiprime, we have

$$0 \neq a_tR^da_t \subseteq LRa_t = Ma_t.$$

Since $Ma_t \neq 0$ and L is d-stable, MI is a nonzero ideal of $R[[y;d]]$ contained in I and it would suffice to show that $(MI)^2 \neq 0$. Therefore, without loss of generality, we may assume that $I \subseteq M[[y;d]]$.

If we are in the characteristic 0 case, then Lemma 9(2) asserts that since $d^t(L) = 0$, we have $d(M) = 0$. In this situation, y commutes with all the coefficients in w. Since R^d is semiprime, there exists $r \in R^d$ such that $a_t r a_t \neq 0$. We now have

$$\begin{aligned} wrw &= (a_t y^t + a_{t+1} y^{t+1} + \cdots) r (a_t y^t + a_{t+1} y^{t+1} + \cdots) \\ &= a_t r a_t y^{2t} + (a_{t+1} r a_t + a_t r a_{t+1}) y^{2t+1} + \cdots \neq 0. \end{aligned}$$

Therefore

$$0 \neq wrw \in I^2,$$

hence $I^2 \neq 0$, as desired.

In light of the above, the only situation left to consider is where $d^t(L) = 0$ and R has characteristic $p > 0$. By Lemma 9(3), $d^{p^{l-1}}(L) = d^{p^{l-1}}(M) = 0$, for $l \geq 1$ such that $p^{l-1} \leq t < p^l$. Therefore, replacing w by $wy^{p^l - t}$ we can now write w as

$$w = a_{p^l} y^{p^l} + a_{p^l+1} y^{p^l+1} + \cdots .$$

However, since $d^{p^l}(M) = 0$, it follows that y^{p^l} commutes with all the coefficients of w. Therefore,

$$w = y^{p^l} w_1 = w_1 y^{p^l},$$

where

$$w_1 = a_{p^l} + a_{p^l+1} y + a_{p^l+2} y^2 + \cdots .$$

Since R^d is semiprime,

$$a_{p^l} A_0 = a_{p^l} R^d \neq 0.$$

Therefore, Lemma 11 tells us that

$$w_1 R w_1 \neq 0.$$

However, y^{p^l} is regular in $R[[y;d]]$, therefore

$$0 \neq y^{p^l}(w_1 R w_1) y^{p^l} = (y^{p^l} w_1) R (w_1 y^{p^l}) = wRw.$$

Hence

$$0 \neq wRw \subseteq I^2$$

and $I^2 \neq 0$, thereby concluding the proof.

□

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