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ACTIONS OF LIE SUPERALGEBRAS ON REDUCED RINGS

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Abstract. In this paper, we look at the question of whether the subring of invariants is always nontrivial when a finite dimensional Hopf algebra acts on a reduced ring. Affirmative answers where given by Kharchenko for group algebras and by Beidar and Grzeszczuk for finite dimensional restricted Lie algebras. Our main result is

Theorem 13 *If R is a graded-reduced ring of characteristic $p > 2$ acted on by a finitely generated restricted K-Lie superalgebra L, then $R^L \neq 0$.*

We can then use Theorem 13 to prove

Corollary 15 *Let R be a reduced algebra over a field K of characteristic $p > 2$ acted on by a finite dimensional restricted K-Lie superalgebra L and let $H = u(L)\#G$, where G is the group of order 2 with the natural action on L. If R^H satisfies a polynomial identity of degree d, then R satisfies a polynomial identity of degree dN, where N is the dimension of H.*

1. Introduction

One of the most significant results in the study of finite groups acting on associative rings is due to Kharchenko [K1], who proved that if a finite group G acts on a reduced ring R, then $A^G \neq 0$, for every nonzero G-stable subring A of R. Over the last twenty years, a great deal of work has taken place examining finite dimensional Hopf algebras and their actions on associative algebras. Looking at Kharchenko's result in terms of Hopf algebras, it provides an affirmative answer, for group algebras $H = KG$, to the following:

Question *If R is a reduced algebra acted on by a finite dimensional Hopf algebra H, is $A^H \neq 0$, for every nonzero H-stable subalgebra A of R?*

In [BG], an affirmative answer is provided by Beidar and Grzeszczuk when $H = u(L)$, the restricted enveloping algebra of a finite dimensional restricted Lie algebra in characteristic p. Observe that group algebras KG and restricted enveloping algebras $u(L)$ are both cocommutative. However, relatively little seems to be known about the action of H on reduced algebras when H is neither commutative nor cocommutative.

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If $L = L_0 \oplus L_1$ is a restricted Lie superalgebra, then we can once again construct the restricted enveloping algebra $u(L)$. Observe that if $L_1 \neq 0$, then $u(L)$ is no longer a Hopf algebra as it is not closed under comultiplication. In this case, L has an automorphism σ or order 2 defined as $\sigma(x) = x$, for all $x \in L_0$, and $\sigma(y) = -y$, for all $y \in L_1$. The action of σ extends to all of $u(L)$ and if we let $G = \{1, \sigma\}$, then we can form the smash product $u(L)\#G$. It turns out that $u(L)\#G$ is a Hopf algebra which is neither commutative nor cocommutative. In an attempt to answer the above question for additional classes of Hopf algebras, especially ones which are neither commutative nor cocommutative, it makes sense to examine the actions of Hopf algebras of the form $H = u(L)\#G$. In this direction, we prove

Corollary 14 *Let R be a reduced algebra over a field K of characteristic $p > 2$ acted on by a finite dimensional restricted K-Lie superalgebra L and let $H = u(L)\#G$, where G is the group of order 2 with the natural action on L. Then $A^H \neq 0$, for every nonzero H-stable subalgebra A of R.*

The final result of this paper combines Corollary 14 with a result of Grzeszczuk and Hryniewicka [GH2, Theorem 4] to prove

Corollary 15 *Let R be a reduced algebra over a field K of characteristic $p > 2$ acted on by a finite dimensional restricted K-Lie superalgebra L and let $H = u(L)\#G$, where G is the group of order 2 with the natural action on L. If R^H satisfies a polynomial identity of degree d, then R satisfies a polynomial identity of degree dN, where N is the dimension of H.*

In $u(L)\#G$, the automorphism σ commutes with elements of L_0 and anti-commutes with elements of L_1. As a result, A^L is G-stable and $A^H = (A^L)^G$. In light of Kharchenko's result, to prove Corollary 14, it suffices to prove that $A^L \neq 0$. Therefore most of the work in this paper will be in extending the result of Beidar and Grzeszczuk from Lie algebras to Lie superalgebras. Observe that as we attempt to prove that $A^L \neq 0$, we can consider A to be a ring which may not contain a multiplicative identity. Therefore, in order to prove the existence of invariants under the actions of Lie superalgebras, it will be more convenient to work with rings which may not contain a unit element. Thus the main result of this paper is stated as

Theorem 13 *If R is a graded-reduced ring of characteristic $p > 2$ acted on by a finitely generated restricted K-Lie superalgebra L, where $K \subseteq C_0$, then $R^L \neq 0$.*

We will now introduce the notation that, unless explicitly stated otherwise, will be used throughout this paper. Along the way, we will see the reasons that some of the hypotheses in Theorem 13 are more general than those in Corollary 14. R will be always be a ring which either has no $\mathbb{Z}$-torsion or has characteristic $p > 2$ and σ will be an automorphism of R such that $\sigma^2 = 1$. Therefore $R = R_0 \oplus R_1$ is $\mathbb{Z}_2$-graded where,

$$R_0 = \{r \in R \mid \sigma(r) = r\} \text{ and } R_0 = \{r \in R \mid \sigma(r) = -r\}.$$

When we say that R is *graded-reduced*, we mean that $R_0 \cup R_1$ contain no nonzero nilpotent elements. We will let Q represent the symmetric Martindale quotient ring of R and C the extended center of R. It follows that the action of σ extends to both Q and C, therefore $Q = Q_0 \oplus Q_1$ and $C = C_0 \oplus C_1$ are also Z_2-graded. Furthermore, since R is graded-reduced, so are Q and C. Observe that since R, Q, and C are graded-reduced, they are all semiprime.

We will make repeated use of the fact that C is von Neumann regular. This means that for every nonzero $\alpha \in C$, there exists some $\beta \in C$ such that $\alpha^2\beta = \alpha$. Thus $e = \alpha\beta$ is a nonzero central idempotent in Q. For any maximal ideal M of C, we can localize Q, R, and C at M

to obtain the rings $\bar{Q}$, $\bar{R}$, and $\bar{C}$, respectively. In this case, $\bar{Q}$ is a centrally closed prime ring and its center is the field $\bar{C}$. An excellent reference for more information about Q and C is the book by Beidar, Martindale, and Mikhalev [BMM]. If $B \subseteq Q$, we say that B is σ-stable if $\sigma(B) \subseteq B$. When $D \subseteq R$ is a σ-stable subring of R, we will denote its symmetric Martindale quotient ring as $Q(D)$.

A function f defined on R is said to be *continuous* if $f(R) \subseteq Q$ and there exists an essential ideal J of R such that $f(J) \subseteq R$. Next, let D_0 be the set of continuous derivations d of R such that $d\sigma = \sigma d$. It now follows that $d(R_0) \subseteq Q_0$ and $d(R_1) \subseteq Q_1$ for all $d \in D_0$. Recall that if δ is a σ-derivation of R. then $\delta(rs) = \delta(r)s + \sigma(r)\delta(s)$, for all $r, s \in R$. We now let D_1 be the set of continuous σ-derivations d of R such that $\delta\sigma = -\sigma\delta$. Thus $\delta(R_0) \subseteq Q_1$ and $\delta(R_1) \subseteq Q_0$, for all $\delta \in D_1$.

Now suppose $\delta_1, \delta_2 \in D_0 \cup D_1$ and let J be an essential ideal of R such that $\delta_1(J), \delta_2(J) \subseteq R$. Since $\delta_i(\sigma(J)) = \sigma(\delta_i(J)) \subseteq \sigma(R) = R$, for $i = 1, 2$, we can replace J by $J + \sigma(J)$ and assume that J is σ-stable. It now follows that J^2 is a σ-stable essential ideal of R such that $\delta_1(\delta_2(J^2)), \delta_2(\delta_1(J^2)) \subseteq R$. Combined with the fact that there are unique extensions of $\sigma, \delta_1, \delta_2$ from R to Q, we see that the compositions $\delta_1\delta_2, \delta_2\delta_1$ are both continuous. Therefore, if we let $Der_\sigma(R, Q) = D_0 \oplus D_1$, then $Der_\sigma(R, Q)$ is a Lie superring. Furthermore, if R has characteristic $p > 2$, $Der_\sigma(R, Q)$ is restricted as $d^p \in D_0$, for all $d \in D_0$. If K is a subring of C_0 such that every element of $Der_\sigma(R, Q)$ is K-linear, then $Der_\sigma(R, Q)$ is a Lie superalgebra over K which is restricted in characteristic p.

We can now say that a K-Lie superalgebra $L = L_0 \oplus L_1$ acts on R if there exists a Lie homomorphism $\phi : L \to Der_\sigma(R, Q)$ such that $K \subseteq C_0$ and the elements of $Der_\sigma(R, Q)$ are K-linear. In addition, if L is restricted, we assume that $\phi(x^{[p]}) = \phi(x)^p$, for all $x \in L_0$, where $[p]$ denotes the pth power map. Although K is a commutative ring which need not be a field, we will assume that 2 is invertible in K.

When no confusion arises, we will often identify the elements of L with their image under ϕ in $Der_\sigma(R, Q)$. For any $B \subseteq Q$, we let $B^\sigma = \{b \in B \mid \sigma(b) = b\}$ and if V is a subset of L, we let $B^V = \{b \in B \mid \delta(b) = 0, \text{ for all } \delta \in V\}$. We also say that B is V-stable if $\delta(B) \subseteq B$, for all $\delta \in V$.

If A is a subring of Q and $B \subseteq Q$, we let $C_A(B) = \{a \in A \mid ab = ba, \text{for all } b \in B\}$ and $Ann_A(B) = \{a \in A \mid ab = 0, \text{for all } b \in B\}$. Since Q is graded-reduced, it is easy to see that if A and B are σ-stable, then $Ann_A(B)$ is a two-sided σ-stable ideal of A. When dealing with elements in $a, b \in Q$, the bracket $[\,,]$ will denote the usual commutator $ab - ba$. However, when dealing with elements of L and $Der_\sigma(R, Q)$, it will denote the multiplication within a Lie superalgebra.

If $d \in D_0$, we say that d is X-inner, if there exists some $q \in Q$ such that $d(r) = qr - rq$, for all $r \in R$. Similarly, if $\delta \in D_1$, we say that δ is X-inner, if there exists some $q \in Q$ such that $\delta(r) = qr - \sigma(r)q$, for all $r \in R$. In addition, we say that σ is X-inner if there exists some nonzero $q \in Q$ such that $qr = \sigma(r)q$, for all $r \in R$. The set of elements of L which can be written as the sum of X-inner elements of L_0 and L_1 is denoted as L_{inn}. Derivations, σ-derivations, and automorphisms which are not X-inner are called X-outer. Two extremely important special cases of Theorem 13 will be where $L_{inn} = L$ and $L_{inn} = 0$. After handling these special cases, much of the proof of Theorem 13 consists of patching these two cases together using mathematical induction and properties of C.

The result of Beidar and Grzeszczuk involves technical difficulties not present in Kharchenko's result. One reason is that when working with the action of a Lie algebra L, in order to split into the X-inner and X-outer cases, one must examine linear combinations over C of elements of L. Observe that although all the elements L may send R to R, linear combinations over C of these elements are continuous derivations. In addition, when restricting the action of L to some subset A of R, it must be checked that the action on A is continuous and the relationship between the extended centers of R and A must also be examined. In this paper, the situation is further complicated because the automorphism σ need not be C-linear. In light of these technical points, in Theorem 13, it is necessary to consider derivations and σ-derivations which are continuous. It is also natural to look at Lie superalgebras over commutative rings K where $K \subseteq C$. Furthermore, it is necessary for K to be contained in C_0, otherwise D_0 and D_1 would not be modules over K.

By twisting the multiplication in a Lie color algebra by a cocyle, one can obtain a Lie superalgebra. Other pointed Hopf algebras are also related to simpler ones in a similar manner. In situations like this, the invariants of a Hopf algebra acting on a ring can sometimes be studied by also twisting the multiplication in the ring, as was done in [BeG]. Observe that if the original ring was reduced, the new ring obtained by twisting would only be graded-reduced. We should point out that graded-reduced rings which are not reduced occur quite naturally. For example, let F be a field and g an automorphism of F of order 2. If we let $G = \{1, g\}$, then the smash product $F\#G$ is certainly graded-reduced but is not reduced as it is isomorphic to the 2×2 matrices over the fixed field F^G. In the hope that Theorem 13 can later be applied to actions of other pointed Hopf algebras, our work has been in the more general setting of graded-reduced rings.

2. The Main Result

At various points in this paper, it will be very useful to restrict the action of L to appropriate ideals or subrings of R. Our first lemma indicates that we can use certain idempotents in C to restrict the action of L to any σ-stable ideal of R.

Lemma 1. *Suppose the K-Lie superalgebra L acts on the graded-reduced ring R.*

1. *If $e = e^2 \in C_0$, then $\delta(e) = 0$, for all $\delta \in L_0 \cup L_1$.*
2. *Let $M \neq 0$ be a σ-stable ideal of R. Then the action of every δ in $L_0 \cup L_1$ restricts to an action on M. Furthermore, there exists some $e = e^2 \in C_0$ that allows us to naturally identify elements of L with elements of eL such that eL is a Lie superalgebra over Ke which acts on M.*
3. *Let $e = e^2 \in C_0$ and let J be a σ-stable essential ideal of R such that $Je \subseteq R$. Then eL is a Lie superalgebra over Ke which acts on Je.*

Proof. For (1), if $\delta \in L_0 \cup L_1$, then since $\sigma(e) = e$, we have

$$\delta(e) = \delta(e^2) = \delta(e)e + e\delta(e) = 2e\delta(e).$$

Multiplying the above equation by e yields $e\delta(e) = 2e\delta(e)$. This immediately implies that $e\delta(e) = 0$, which in turns implies that $\delta(e) = 0$.

For part (2), let Q and $Q(M)$ represent the symmetric Martindale quotient rings of R and M, respectively. To show that δ acts on M, we need to show that $\delta(M) \subseteq Q(M)$ and also that δ sends some essential ideal of M into M. Observe that there exists some $e = e^2 \in C$ such that $Q(M) = Qe$. Since M is σ-stable, it follows that $\sigma(e) = e$. If $r \in M$ and $\delta \in L_0 \cup L_1$ then $r = qe$, for some $q \in Q$, and

$$\delta(r) = \delta(qe) = \delta(q)e \in Qe = Q(M).$$

Thus $\delta(M) \subseteq Q(M)$.

Next, let I be an essential ideal of R such that $\delta(I) \subseteq R$. Since $\delta(\sigma(I)) = \sigma(\delta(I)) \subseteq \sigma(R) = R$, we can choose I to be σ-stable. If we let $P = I \cap M$, then P^2 is a σ-stable essential ideal of M and

$$\delta(P^2) \subseteq \delta(P)P + P\delta(P) \subseteq RM + MR \subseteq M.$$

Combining the facts that $\delta(M) \subseteq Q(M)$ and $\delta(P^2) \subseteq M$, we see that δ does act on M.

Now if $r \in M$ and $\delta \in L_0 \cup L_1$ then $r = qe$, for some $q \in Q$, and

$$e\delta(r) = e\delta(qe) = e\delta(q)e = \delta(q)e = \delta(r).$$

The previous equation indicates that we can naturally identify the actions of $e\delta$ and δ on M. Since $K \subseteq C_0$, we have $Ke \subseteq C_0e = (Ce)_0$, thus Ke is contained in the even part of the extended center of M. Therefore, if we let $eL = \{e\delta \mid \delta \in L\}$, then eL is a Ke-Lie superalgebra acting on M.

For part (3), let $M = Je$; since J is essential in R, we have $Q(M) = Q(J)e = Q(R)e = Qe$. We can now apply part (2) to the σ-stable ideal M. □

Our next lemma will allow us to reduce the actions of L and L_0 to some naturally occuring subrings of R and R_0.

Lemma 2. *Let R be graded-reduced and let T be a σ-stable subring of Q which contains C_0 such $T \cap J \neq 0$, for every σ-stable ideal $J \neq 0$ of R.*

1. *T embeds naturally in $Q(T \cap R)$ and T_0 embeds naturally in $Q((T \cap R)_0)$.*
2. *If the K-Lie superalgebra L acts on R such that T is L-stable, then L acts on $T \cap R$ and L_0 acts on $(T \cap R)_0$.*

Proof. For (1), let $t \in T$. Since $t, \sigma(t) \in Q$, there exists an essential ideal J of R such that $Jt + tJ + J\sigma(t) + \sigma(t)J \subseteq R$. Therefore

$$\sigma(J)t + t\sigma(J) = \sigma(J\sigma(t) + \sigma(t)J) \subseteq \sigma(R) = R.$$

As a result, if let $M = J + \sigma(J)$, then M is a σ-stable essential ideal of R such that $Mt + tM \subseteq R$.

Let $U = Ann_R((T \cap M)_0)$; if we can show that $U = 0$, then we will have shown that $T \cap M$ is an essential ideal of $T \cap R$ and also that $(T \cap M)_0$ is an essential ideal of $(T \cap R)_0$. If $U \neq 0$, then UM is a non-zero σ-stable of R. Hence $T \cap UM$ is nonzero graded-reduced subring of R, thus $(T \cap UM)_0 \neq 0$. As a result, if $0 \neq v \in (T \cap UM)_0$, then

$$0 \neq v^2 \in U \cdot (T \cap M)_0 = Ann_R((T \cap M)_0) \cdot (T \cap M)_0 = 0,$$

a contradiction. Thus $U = 0$ and $T \cap M$ and $(T \cap M)_0$ are essential ideal of $T \cap R$ and $(T \cap R)_0$, respectively. Since

$$(T \cap M)t + t(T \cap M) \subseteq T \cap R,$$

t sends an essential ideal of $T \cap R$ into $T \cap R$ by both left and right multiplication. Thus $t \in Q(T \cap R)$ and so, T embeds in $Q(T \cap R)$.

In addition, if $t \in T_0$, then it is clear that

$$(T \cap M)_0 t + t(T \cap M)_0 \subseteq (T \cap R)_0.$$

Since $(T \cap M)_0$ is essential in $(T \cap R)_0$, $t \in Q((T \cap R)_0)$. Hence T_0 embeds in $Q((T \cap R)_0)$.

For (2), let $\delta \in L_0 \cup L_1$. Since $\delta(T) \subseteq T$, part (1) implies that

$$\delta(T \cap R) \subseteq T \subseteq Q(T \cap R).$$

Next, let J be an essential ideal of R such that $\delta(J) \subseteq R$. Observe that

$$\delta(\sigma(J)) = \sigma(\delta(J)) \subseteq \sigma(R) = R.$$

Therefore, if we let $M = J + \sigma(J)$, then M is a σ-stable essential ideal of R such that $\delta(M) \subseteq R$. The argument in part (1) shows that $T \cap M$ is an essential ideal of $T \cap R$. Moreover,

$$\delta(T \cap M) \subseteq T \cap R.$$

Thus $T \cap M$ is an essential ideal of $T \cap R$ which is sent by δ into $T \cap R$. Since $\delta(T \cap R) \subseteq Q(T \cap R)$ and $\delta(T \cap M) \subseteq T \cap R$, we see that δ acts on $T \cap R$.

Since T contains C_0, it follows from part (1) that C_0 embeds in the extended center of $T \cap R$. Thus K embeds in the even part of the extended center of $T \cap R$ and the K-Lie superalgebra L does act on $T \cap R$.

If $\delta \in L_0$ then the facts that $\delta(R_0) \subseteq Q_0$ and $\delta(T) \subseteq T$ combine with part (1) to imply that

$$\delta((T \cap R)_0) \subseteq T_0 \subseteq Q((T \cap R)_0).$$

Since

$$\delta((T \cap M)_0) \subseteq (T \cap R)_0,$$

we see that $(T \cap M)_0$ is an essential ideal of $(T \cap R)_0$ which is sent by δ into $(T \cap R)_0$

Finally, since T_0 contains C_0, it follows from part (1) that C_0 embeds in the extended center of $(T \cap R)_0$. Thus K embeds in the even part of the extended center of $(T \cap R)_0$ and the K-Lie algebra L_0 acts on $(T \cap R)_0$. □

Our next lemma examines the nature of those elements of Q which induce the X-inner elements of $L_0 \cup L_1$.

Lemma 3. *Let R be a graded-reduced ring acted on by the K-Lie superalgebra L.*

1. *If $\delta \in L_i$ is X-inner, then δ is induced by some $a \in Q_i$.*
2. *If $\delta, d \in L_0 \cup L_1$ and d is induced by the homogeneous element $a \in Q$, then $[\delta, d]$ is induced by $\delta(a)$. Thus L_{inn} is an ideal of L.*
3. *If $a, b \in Q_0$ induce the same derivation of R, then $a - b \in C_0$.*
4. *If $a, b \in Q_1$ induce the same σ-derivation of R, then $a = b$.*
5. *If $a, b \in Q_1$ induce the σ-derivations d, δ, then $ab + ba$ induces the derivation $[\delta, d]$.*

Proof. For (1), first suppose $\delta \in L_0$ is induced by $a = a_0 + a_1 \in Q$. Then

$$\delta(r) = ar - ra = (a_0 + a_1)r - r(a_0 + a_1) \tag{2.1}$$

$$= (a_0 r - r a_0) + (a_1 r - r a_1).$$

This immediately implies that

$$\delta(\sigma(r)) = (a_0\sigma(r) - \sigma(r)a_0) + (a_1\sigma(r) - \sigma(r)a_1)$$

and

$$\sigma(\delta(r)) = (a_0\sigma(r) - \sigma(r)a_0) - (a_1\sigma(r) - \sigma(r)a_1).$$

Since δ and σ commute, the previous equations imply that $2(a_1\sigma(r) - \sigma(r)a_1) = 0$, hence $a_1\sigma(r) - \sigma(r)a_1 = 0$. Replacing r by $\sigma(r)$ yields $a_1 r - r a_1 = 0$. Equation 2.1 now becomes

$$\delta(r) = (a_0 r - r a_0) + (a_1 r - r a_1) = a_0 r - r a_0.$$

Thus δ is induced by $a = a_0 \in Q_0$.

Now suppose $\delta \in L_1$ is induced by $a = a_0 + a_1 \in Q$. Then

$$\begin{aligned}(2.2) \qquad \delta(r) &= ar - \sigma(r)a = (a_0 + a_1)r - \sigma(r)(a_0 + a_1)\\ &= (a_0 r - \sigma(r)a_0) + (a_1 r - \sigma(r)a_1).\end{aligned}$$

This implies that

$$\delta(\sigma(r)) = (a_0\sigma(r) - ra_0) + (a_1\sigma(r) - ra_1)$$

and

$$\sigma(\delta(r)) = (a_0\sigma(r) - ra_0) - (a_1\sigma(r) - ra_1).$$

Since $\delta\sigma = -\sigma\delta$, adding the previous equations results in $2(a_0\sigma(r) - ra_0) = 0$, hence $a_0\sigma(r) - ra_0 = 0$. Replacing r by $\sigma(r)$ yields $a_0 r - \sigma(r)a_0 = 0$. Equation 2.2 now becomes

$$\delta(r) = (a_0 r - \sigma(r)a_0) + (a_1 r - \sigma(r)a_1) = a_1 r - \sigma(r)a_1.$$

Thus δ is induced by $a = a_1 \in Q_1$.

For (2), we will need to examine four cases and will begin by supposing that $d(r) = ar - g(r)a$ is a g-derivation and δ is an h-derivation, where g, h are commuting automorphisms of R.

Therefore

$$\begin{aligned}(2.3) \qquad [\delta, d](r) = \delta(ar - g(r)a) - (a\delta(r) - g(\delta(r))a) =\\ \delta(a)r + h(a)\delta(r) - \delta(g(r))a - h(g(r))\delta(a) - a\delta(r) + g(\delta(r))a.\end{aligned}$$

For the first case, suppose $d, \delta \in L_0$; then $g = h = 1$ and $h(a) = a$. Then 2.3 simplifies to $[\delta, d](r) = \delta(a)r - r\delta(a)$. Hence $[\delta, d]$ is X-inner and induced by $\delta(a)$.

Second, suppose $d \in L_0$ and $\delta \in L_1$; then $g = 1, h = \sigma, h(a) = a$, and $g\delta = \delta g$. In this case, 2.3 simplifies to $[\delta, d](r) = \delta(a)r - \sigma(r)\delta(a)$. Hence, once again, $[\delta, d]$ is X-inner and induced by $\delta(a)$.

For our third case, suppose $d \in L_1$ and $\delta \in L_0$; then $g = \sigma, h = 1, h(a) = a$, and $g\delta = \delta g$. This time 2.3 simplifies to $[\delta, d](r) = \delta(a)r - \sigma(r)\delta(a)$. As before, $[\delta, d]$ is X-inner and induced by $\delta(a)$.

Finally, suppose $d, \delta \in L_1$. Then, instead of looking at 2.3, we need to examine

$$(2.4) \qquad (\delta d + d\delta)(r) = \delta(a)r + h(a)\delta(r) - \delta(g(r))a - h(g(r))\delta(a) + a\delta(r) - g(\delta(r))a.$$

In this case, $g = h = \sigma$, $h(a) = -a$, and $g\delta = -\delta g$. Therefore 2.4 now becomes $[\delta, d](r) = \delta(a)r - r\delta(a)$ and so, $[\delta, d]$ is X-inner and induced by $\delta(a)$. As a result, in every case $[\delta, d]$ is induced by $\delta(a)$. Thus $[L, L_{inn}] \subseteq L_{inn}$ and so, L_{inn} is an ideal of L.

For (3), if $a, b \in Q_0$ both induce the derivation δ, then we have

$$0 = \delta(r) - \delta(r) = (ar - ra) - (br - rb) = (a - b)r - r(a - b),$$

for all $r \in R$. Since $a - b$ commutes with every element of R, it also commutes with every element of Q. Hence $a - b \in C \cap Q_0 = C_0$.

For (4), if $a, b \in Q_1$ both induce the skew derivation δ, then we have

$$0 = \delta(r) - \delta(r) = (ar - \sigma(r)a) - (br - \sigma(r)b) = (a - b)r - \sigma(r)(a - b),$$

for all $r \in R$. If we let $q = a - b$, then $q \in Q_1$ and the previous equation tells us that $qr = \sigma(r)q$, for all $r \in R$. But implies that $qr = \sigma(r)q$, for all $r \in Q$. If we now let $r = q$ in the equation $qr = \sigma(r)q$ and use the fact that $\sigma(q) = -q$, we obtain $q^2 = -q^2$. Hence $2q^2 = 0$ and so, $q^2 = 0$. However q is homogeneous and Q contains no nonzero nilpotent homogeneous elements, hence $q = 0$. Thus $0 = q = a - b$, hence $a = b$.

For (5), if d is induced by $a \in Q_1$, then it follows from part (2) that $[\delta, d]$ is induced by $\delta(a)$. But since δ is induced by b, we have $\delta(a) = ba - \sigma(a)b = ab + ba$. Thus $[\delta, d]$ is induced by $ab + ba$, as desired. □

Our next lemma is essentially a restatement of Theorem 2.6 and Corollary 2.7 of [BG]. It will be extremely useful in analyzing the special cases where all the elements of either L_0 or L_1 are X-inner.

Lemma 4. *Let R be a semiprime ring and let A be a semiprime subring of Q which contains C and is finitely generated as a C-module. If we let $T = C_Q(A)$, then $T \cap J \neq 0$, for every nonzero ideal J of R.*

Proof. We will first show that $T \cap R \neq 0$. Since A is finitely generated over C, there exist $a_1, \dots, a_n \in A$ such that $A = Ca_1 + \cdots + Ca_n$. Let M be a maximal ideal of C such that the image $\bar{A}$ of A under the localization at M is nonzero. Let $\bar{Q}$, $\bar{C}$ denote, respectively, the localizations of Q and C at M. Recall that $\bar{C}$ is a field and is the center of the centrally closed prime ring $\bar{Q}$.

It is clear that $\bar{A} = \bar{C}\bar{a}_1 + \cdots + \bar{C}\bar{a}_n$. Furthermore, Proposition 5(i) of [GH1] asserts that $\bar{A}$ is also semiprime. Thus $\bar{A}$ is a semisimple algebra which is finite dimensional over the field $\bar{C}$. We can now use some of the arguments from the proof of Theorem 2.6 in [BG].

Observe that $\bar{A}$ is a Frobenius algebra over $\bar{C}$. Therefore there exists a non-degenerate associate bilinear form $F : \bar{A} \times \bar{A} \to \bar{C}$. We can fix a basis $\{\bar{e}_1, \dots, \bar{e}_m\}$ for $\bar{A}$ and let $\{\bar{e}_1^*, \dots, \bar{e}_m^*\}$ be the dual basis, where $m \leq n$. Thus $F(\bar{e}_i, \bar{e}_j^*) = 1$ when $i = j$ and $F(\bar{e}_i, \bar{e}_j^*) = 0$ when $i \neq j$.

For $1 \leq t \leq n$, there exist $\bar{\alpha}_{tkj}, \bar{\beta}_{tjk} \in \bar{C}$ such that

$$\bar{a}_t \bar{e}_k^* = \sum_{j=1}^{m} \bar{\alpha}_{tkj} \bar{e}_j^* \quad \text{and} \quad \bar{e}_j \bar{a}_t = \sum_{k=1}^{m} \bar{\beta}_{tjk} \bar{e}_k. \tag{2.5}$$

Since F is associative, we have

$$\bar{\alpha}_{tkj} = F(\bar{e}_j, \bar{a}_t \bar{e}_k^*) = F(\bar{e}_j \bar{a}_t, \bar{e}_k^*) = \bar{\beta}_{tjk}. \tag{2.6}$$

Let $\xi \in C \backslash M$ be a common denominator for every $\bar{\alpha}_{tkj}, \bar{e}_k^*, \bar{e}_j^*$. Thus

$$\bar{\alpha}_{tkj} = \xi^{-1} \alpha_{tkj}, \quad \bar{e}_k^* = \xi^{-1} e_k^*, \quad \text{and} \quad \bar{e}_j = \xi^{-1} e_j.$$

Rewriting the last three equations in Q, there exists some $\eta \in C \backslash M$ such that

$$\eta\xi \bar{\alpha}_{tkj} = \eta \alpha_{tkj}, \quad \eta\xi \bar{e}_k^* = \eta e_k^*, \quad \text{and} \quad \eta\xi \bar{e}_j = \eta e_j.$$

Using these facts along with 2.6, we can rewrite 2.5 as

$$\eta\xi a_t e_k^* = \sum_{j=1}^{m} \eta \alpha_{tkj} e_j^* \quad \text{and} \quad \eta\xi e_j a_t = \sum_{k=1}^{m} \eta \alpha_{tkj} e_k. \tag{2.7}$$

We can now define the map $\rho : Q \to Q$ as $\rho(x) = \eta\xi \sum_{k=1}^{m} e_k^* x e_k$, for all $x \in Q$. If $1 \leq t \leq n$ and $x \in Q$, by applying 2.7, we have

$$\rho(x) a_t = (\eta\xi \sum_{j=1}^{m} e_j^* x e_j) a_t = \sum_{j=1}^{m} e_j^* x (\eta\xi e_j a_t) = \sum_{j=1}^{m} e_j^* x \sum_{k=1}^{m} \eta \alpha_{tkj} e_k =$$

$$\sum_{j=1}^{m}\sum_{k=1}^{m}\eta\alpha_{tkj}e_j^*xe_k = \sum_{k=1}^{m}\left(\sum_{j=1}^{m}\eta\alpha_{tkj}e_j^*\right)xe_k = \sum_{k=1}^{m}\eta\xi a_t e_k^* x e_k =$$

$$a_t(\eta\xi\sum_{k=1}^{m}e_k^*xe_k) = a_t\rho(x).$$

Since $\{a_1,\dots,a_n\}$ spans A over C, it is clear that $\rho(x)a = a\rho(x)$ for all $x \in Q$ and $a \in A$. Thus $\rho(Q) \subseteq T$. Martindale's Theorem asserts that since $\{\bar{e}_1,\dots,\bar{e}_m\}$ and $\{\bar{e}_1^*,\dots,\bar{e}_m^*\}$ are subsets of the prime ring $\bar{Q}$ which are linearly independent over $\bar{C}$, there exists some $q \in Q$ such that $\sum_{k=1}^{m}\bar{e}_k^*\bar{q}\bar{e}_k \neq 0$ in $\bar{Q}$. Since $\eta,\xi \in C\backslash M$, both $\bar{\eta}$ and $\bar{\xi}$ are invertible in $\bar{Q}$. Therefore $\bar{\eta}\bar{\xi}\sum_{k=1}^{m}\bar{e}_k^*\bar{q}\bar{e}_k \neq 0$ in $\bar{Q}$. But this immediately implies that $\rho(q) = \eta\xi\sum_{k=1}^{m}e_k^*qe_k \neq 0$ in Q. Hence $\rho(Q) \neq 0$.

Since ρ involves only a finite number of elements of Q, there exists an essential ideal I of R such that $\rho(I) \subseteq R$. Observe that since I is essential, it has the same symmetric Martindale quotient ring as R. Thus $Q(I) = Q$, which immediately implies that $\rho(Q(I)) \neq 0$. Viewing I as a semiprime ring, we can apply a result of Beidar [B] which asserts that since ρ does not vanish on $Q(I)$, then it also does not vanish on I. Combining several of our observations, we can now conclude that

$$0 \neq \rho(I) \subseteq T \cap R = C_R(A).$$

Thus we have succeeded in showing that $T \cap R \neq 0$.

To complete the proof, suppose J is a nonzero ideal of R. Then there exists some $e = e^2 \in C$ such that $Q(J) = Qe$. We are now in the situation where J is a semiprime ring with symmetric Martindale quotient ring Qe and extended center Ce such that Ae is a semiprime subring of Qe which contains Ce and is finitely generated as a Ce-module. Therefore we can apply our previous argument to conclude that $C_J(eA) \neq 0$. Since multiplication by $1-e$ annihilates J, it follows that if $r \in C_J(eA)$ and $a \in A$, then

$$[r,a] = [r,((1-e)+e)a] = [r,(1-e)a] + [r,ea] = 0+0 = 0.$$

Hence $r \in C_J(A)$. Thus $0 \neq C_J(eA) \subseteq C_J(A) = T \cap J$, thereby concluding the proof. ☐

We can now prove the important special cases of our main result where either $L_0 \subseteq L_{inn}$ or $L_1 \subseteq L_{inn}$. We have stated the following lemma in such a way that we will be able to apply it in both the characteristic $p > 2$ and characteristic 0 cases.

Lemma 5. *Suppose the graded-reduced ring R is acted on by the finitely generated K-Lie superalgebra L such that the elements of L_0 act on R as algebraic derivations. If either $L_0 \subseteq L_{inn}$ or $L_1 \subseteq L_{inn}$, then $R^L \neq 0$.*

Proof. Let V be an ideal of L which is finitely generated over K and contained in L_{inn}. We can write

$$V = (Kx_1 + \cdots + Kx_s) \oplus (Ky_1 + \cdots + Ky_t),$$

where $x_i \in L_0 \cap L_{inn}$ and $y_j \in L_1 \cap L_{inn}$, for all i,j. By Lemma 3(1), for every i,j, we can let $a_i \in Q_0$ induce the derivation corresponding to x_i and let $b_j \in Q_1$ induce the σ-derivation corresponding to y_j. Next, let W_0 and W_1 be the span over C of the sets $\{1,a_1,\dots,a_s\}$ and $\{b_1,\dots,b_t\}$, respectively. Then Lemma 3(4)(5) implies that $W = W_0 \oplus W_1$ is a finitely generated Lie superalgebra over C. Now let A be the subalgebra of Q generated over C by the set $\{1,a_1,\dots,a_s\} \cup \{b_1,\dots,b_t\}$. Observe that A is spanned over C by Poincaré-Birkhoff-Witt monomials of the form ${a_1}^{i_1}\cdots{a_s}^{i_s}{b_1}^{j_1}\cdots{b_t}^{j_t}$, where each $j_l \leq 1$. Since each a_i induces a derivation which is algebraic over C, Proposition 2.2 of [BG] asserts that each a_i is algebraic over C. As a result, we can bound the exponents of the a_i which appear in the monomials which span A over C. Hence A is finitely generated over C.

Since R and Q are graded-reduced and A is σ-stable, we see that A is semiprime. Next, let $T = Q^V$ and let $S = C_Q(A)$. If $J \neq 0$ is a σ-stable ideal of R, then Lemma 4 implies that $S \cap J \neq 0$. Since $S \cap J$ is σ-stable, it follows that $(S \cap J)_0 \neq 0$. However, it is easy to see that $(S \cap J)_0 \subseteq T \cap J$. As a result, T is a σ-stable subring of Q which contains C_0 such $T \cap J \neq 0$, for every σ-stable ideal $J \neq 0$ of R. Furthermore, since V is an ideal of L, T is also L-stable. Thus Lemma 2(2) now tells us that L acts on $T \cap R$. Since $T \cap R = R^V$, we now know that L acts on the nonzero graded-reduced ring R^V.

We will first consider the case where $L_0 \subseteq L_{inn}$. In this case, let

$$V = L_0 \oplus [L_0, L_1].$$

Observe that V is certainly an ideal of L which is finitely generated over K and Lemma 3(2) implies that $V \subseteq L_{inn}$. Therefore we can apply our argument above to assert that $R^V \neq 0$ and L acts on R^V. Since V contains L_0, the action of L on R^V is quite special. In particular, if $r \in R^V$, $d \in L_0$, and $\delta, \delta_1 \in L_1$, then we have

$$d(r) = 0, \quad \delta^2(r) = 0, \quad \text{and} \quad \delta(\delta_1(r)) = -\delta_1(\delta(r)). \tag{2.8}$$

There exists a σ-stable essential ideal I of R^V such that $\delta(I) \subset R^V$, for all t of the σ-derivations that generate L_1 over K. It follows from 2.8 that the composition of any $t+1$ elements of L vanishes on R^V, therefore if we let $J = I^t$, then any composition of the generators of L_1 must send J into R^V. If L_1 vanishes on J, then clearly $J \subseteq R^L$ and we are done in this case. However, if L_1 does not vanish on J, then there exists a smallest positive integer m such the product of any $m+1$ elements of L_1 vanishes on J but there exists a composition ρ of m generators of L_1 such that $0 \neq \rho(J) \subseteq R^V$. Therefore, if $\delta \in L_0 \cup L_1$, then $\delta(\rho(J)) = 0$. Thus

$$0 \neq \rho(J) \subseteq R^L$$

and so, $R^L \neq 0$.

Finally, let us consider the case where $L_1 \subseteq L_{inn}$. In this case, let

$$V = [L_1, L_1] \oplus L_1.$$

Once again, V is an ideal of L which is finitely generated over K and Lemma 3(2) implies that $V \subseteq L_{inn}$. Therefore our previous argument again implies that $R^V \neq 0$ and L acts on R^V. However, in this case, Lemma 2(2) also asserts that L_0 acts on the nonzero reduced ring $(R^V)_0$. Since L_0 is a finitely generated K-Lie algebra of algebraic derivations, Theorems 3.4 and 3.5 of [BG] imply that $((R^V)_0)^{L_0} \neq 0$. Clearly $((R^V)_0)^{L_0} \subseteq R^L$, thus $R^L \neq 0$, thereby concluding the proof. □

In the main result of this paper, L will be finitely generated and restricted. The concept of being restricted is only valid in characteristic p and we can observe that L being finitely generated and restricted is equivalent to L being finitely generated with L_0 consisting solely of algebraic derivations. In light of this, there is a natural characteristic 0 analog of our main result. In fact, the characteristic 0 version is much easier and, as we will now see, follows directly from Lemma 5.

Theorem 6. *Let R be a graded-reduced ring of characteristic 0 acted on by a finitely generated K-Lie superalgebra L, where $K \subseteq C_0$. If every derivation in L_0 is algebraic over C, then $R^L \neq 0$.*

Proof. Since every derivation in L_0 is algebraic over C and the characteristic is 0, it follows from [K2] that every element of L_0 is X-inner. Therefore $L_0 \subseteq L_{inn}$ and the result follows immediately from Lemma 5. □

Our next lemma deals with three more important special cases of our main result.

Lemma 7. *Let R be a graded-reduced ring acted on by a finitely generated restricted K-Lie superalgebra L, where $K \subseteq C_0$. Suppose at least one of the following three conditions holds:*

1. *there exists some nonzero $a \in Q$ such that $ar = \sigma(r)a$, for all $r \in Q$;*
2. *σ is not C-linear;*
3. *some element of L_1 is not C-linear.*

Then $R^L \neq 0$.

Proof. We begin with an observation that will be used in the proofs of parts (1) and (2). Since C is von Neumann regular, if $0 \neq \beta \in C_0$, then there exists $\gamma_1 \in C$ such that $\beta^2\gamma_1 = \beta$. Applying σ to both sides of this equation yields $\beta^2\sigma(\gamma_1) = \beta$. Adding our last two equations and dividing by 2 yields $\beta^2(\frac{\gamma_1+\sigma(\gamma_1)}{2}) = \beta$. Therefore if we let $\gamma = \frac{\gamma_1+\sigma(\gamma_1)}{2}$, then $\gamma \in C_0$ and $\beta^2\gamma = \beta$. If we let $e = \beta\gamma$, then it is clear that $0 \neq e = e^2 \in C_0$. In addition, since $e \in Q\beta$ and $\beta = \beta e \in Qe$, it follows that $Q\beta = Qe$. Hence β is invertible in Qe. Thus, for every $0 \neq \beta \in C_0$, there exists some $e = e^2 \in C_0$ such that β is invertible in Qe.

Suppose condition (1) holds. We can write $a = a_0 + a_1$, where $a_0 \in Q_0$ and $a_1 \in Q_1$. Then if we let $r = a_1$ in the equation $ar = \sigma(r)a$, we obtain

$$(a_0 + a_1)a_1 = -a_1(a_0 + a_1).$$

Looking at the even part of each side of the previous equation tells us that $2{a_1}^2 = 0$, which implies that $a_1 = 0$. Hence $a = a_0 \in Q_0$.

It is easy to see that $a^2r = \sigma^2(r)a^2 = ra^2$, for all $r \in Q$. Thus a^2 is a nonzero element of C_0. Our observation above tells us that there exists some $e = e^2 \in C_0$ such that a^2 is invertible in Qe. Thus ae is an invertible element of Qe. Next, let J be a σ-stable essential ideal of R such that $Je \subseteq R$. By Lemma 1(3), eL acts on Je and it suffices to show that $(Je)^{eL} \neq 0$. However, in $Q(Je) = Qe$, the element ea is now invertible. Therefore, in order to prove that $R^L \neq 0$, we may reduce to the case where a is invertible.

In light of our reduction, we are now in the situation where $\sigma(r) = ara^{-1}$, for all $r \in Q$. If $\delta \in L_1$, then δ and σ anti-commute. Therefore, if $r \in Q$, then using the fact that $\delta(a^{-1}) = -a^{-1}\delta(a)a^{-1}$, we obtain

$$0 = (\delta\sigma + \sigma\delta)(r) = \delta(\sigma(r)) + \sigma(\delta(r)) = \delta(ara^{-1}) + a\delta(r)a^{-1} =$$
$$\delta(a)ra^{-1} + a\delta(r)a^{-1} + a\sigma(r)\delta(a^{-1}) + a\delta(r)a^{-1} =$$
$$\delta(a)ra^{-1} + a\delta(r)a^{-1} - a\sigma(r)a^{-1}\delta(a)a^{-1} + a\delta(r)a^{-1}.$$

Multiplying the previous equation by a^{-1} on the left and a on the right yields

$$0 = a^{-1}\delta(a)r + \delta(r) - \sigma(r)a^{-1}\delta(a) + \delta(r).$$

But this can easily be rewritten as

$$2\delta(r) = -a^{-1}\delta(a)r + \sigma(r)a^{-1}\delta(a).$$

Dividing both sides by $\frac{1}{2}$ and letting $b = \frac{-a^{-1}\delta(a)}{2}$ gives us

$$\delta(r) = br - \sigma(r)b.$$

Thus every element of L_1 is X-inner. Lemma 5 now implies that $R^L \neq 0$.

Now suppose condition (2) holds. Then there exists some nonzero $\alpha \in C$ such that $\sigma(\alpha) = -\alpha$. Since α^2 is a nonzero element of C_0, the argument used in the proof of part (1) tells us that there is some $e = e^2 \in C$ such that α is invertible in Qe. Using the same reduction as in part (1), we may now reduce down to the case where α is invertible.

If $\delta \in L_1$ and $r \in Q$, we have

$$\delta(\alpha r) = \delta(\alpha)r - \alpha\delta(r)$$

and

$$\delta(r\alpha) = \delta(r)\alpha + \sigma(r)\delta(\alpha).$$

Since $\delta(\alpha r) = \delta(r\alpha)$, the previous equations imply that

$$\delta(\alpha)r - \alpha\delta(r) = \delta(r)\alpha + \sigma(r)\delta(\alpha).$$

However $\alpha \in C$, therefore we can simplify further to obtain

$$2\alpha\delta(r) = \delta(\alpha)r - \sigma(r)\delta(\alpha).$$

We are in the situation where α is invertible, therefore if we let $b = \frac{\alpha^{-1}\delta(\alpha)}{2}$, we can rewrite the previous equation as

$$\delta(r) = br - \sigma(r)b.$$

Thus we are once again in the situation where every element of L_1 is X-inner. Therefore, Lemma 5 again implies that $R^L \neq 0$.

Finally, let us suppose that condition (3) holds. In light of part (2), we may assume that σ is C-linear. In this case, there exists some $\delta \in L_1$ and $c \in C$ such that $\delta(c) \neq 0$. If $r \in Q$, using the fact that $\sigma(c) = c$, we have

$$\delta(cr) = \delta(c)r + c\delta(r)$$

and

$$\delta(rc) = \delta(r)c + \sigma(r)\delta(c).$$

Since $\delta(cr) = \delta(rc)$ and $c\delta(r) = \delta(r)c$, the previous equations imply that

$$\delta(c)r = \sigma(r)\delta(c).$$

However, since $\delta(c) \neq 0$, it follows from part (1) that $R^L \neq 0$, thereby concluding the proof. □

In [BM, Proposition 1.1], Bergen and Montgomery showed that a continuous derivation d of a prime ring R is X-inner if and only if there exist $a, b \in R$, with $a \neq 0$, such that $ad(ra) = bra - arb$, for all $r \in R$. To handle the case where $L_{inn} = 0$, we will not only use the result in [BM] but we will also need to extend it to continuous σ-derivations which are induced by odd elements in Lie superalgebras. We should point out that Kharchenko [K3] has proven more general results on the nature of X-inner derivations and skew derivations but we will not need the full generality of his results.

Proposition 8. *Let R be a prime ring and δ a continuous σ-derivation of R where $\delta\sigma = -\sigma\delta$ and $\sigma^2 = 1$. Then δ is X-inner if and only if there exists some $a, b \in R$ such that $a \neq 0$, $\sigma(a) = a$, and $a\delta(ra) = bra - a\sigma(r)b$, for all $r \in R$.*

Proof. In one direction, since δ is X-inner, there exist some $q \in Q$ such that

$$\delta(r) = qr - \sigma(r)q,$$

for all $r \in R$. Next, let $I \neq 0$ be an ideal of R such that $Iq + qI \subseteq R$. Since R is prime, we can replace I by $I \cap \sigma(I)$ and can therefore assume that I is σ-stable. Thus there exists some nonzero $a \in I$ such that $\sigma(a) = a$. If we multiply the equation $\delta(r) = qr - \sigma(r)q$ on the left by a and then replace r by ra, we obtain

$$a\delta(ra) = a(q(ra) - \sigma(ra)q) = (aq)ra - a\sigma(r)(aq).$$

Letting $b = aq$ results in

$$a\delta(ra) = bra - a\sigma(r)b,$$

as desired.

In the other direction, suppose $a\delta(ra) = bra - a\sigma(r)b$, for all $r \in Q$, where $a, b \in R$ such that $a \neq 0$ and $\sigma(a) = a$. Let J be a nonzero σ-stable ideal of R such that $\delta(J) \subseteq R$ and define the left module map q from RaJ to R as

$$(xa\sigma(y))q = xby - xa\delta(y),$$

for all $x \in R$ and $y \in J$. In order to show that $q \in Q$, we first need to check that q is well-defined. Thus we must show that whenever

$$\sum_i x_i a\sigma(y_i) = 0,$$

where $x_i \in R$ and $y_i \in J$, then it follows that such that

$$\sum_i x_i b y_i - x_i a\delta(y_i) = 0.$$

To this end, suppose $x_i \in R$ and $y_i \in J$ such $\sum_i x_i a\sigma(y_i) = 0$ and let $r \in R$. Using the facts that

$$by_i ra = a\sigma(y_i r)b + a\delta((y_i r)a) \quad \text{and} \quad \delta(y_i)ra = \delta(y_i(ra)) - \sigma(y_i)\delta(ra),$$

we obtain

$$(\sum_i x_i b y_i - x_i a\delta(y_i))ra = \sum_i x_i b(y_i r)a - x_i a\delta(y_i)ra =$$

$$\sum_i x_i a\sigma(y_i r)b + x_i a\delta((y_i r)a) - x_i a\delta(y_i(ra)) + x_i a\sigma(y_i)\delta(ra) =$$

$$(\sum_i x_i a\sigma(y_i))(\sigma(r)b + \delta(ra)) = 0.$$

Thus we have shown that if

$$\sum_i x_i a\sigma(y_i) = 0,$$

then

$$(\sum_i x_i b y_i - x_i a\delta(y_i))Ra = 0.$$

Since R is prime, this implies that $\sum_i x_i b y_i - x_i a\delta(y_i) = 0$, as desired. Hence q is well-defined.

Next, we need to show that q does induce δ. If we let $x, r \in R$ and $y \in J$, then by using the definition of q, we obtain

$$(xa\sigma(y))(qr - \sigma(r)q - \delta(r)) = (xa\sigma(y)q)r - xa\sigma(yr)q - xa\sigma(y)\delta(r) =$$

$$xbyr - xa\delta(y)r - xbyr + xa\delta(yr) - xa\sigma(y)\delta(r) = 0.$$

Hence

$$RaJ(qr - \sigma(r)q - \delta(r)) = 0$$

and the fact that R is prime implies that

$$qr - \sigma(r)q - \delta(r) = 0.$$

Therefore $\delta(r) = qr - \sigma(r)q$, for all $r \in R$.

Finally, let I be a nonzero σ-stable ideal of R which is contained in RaJ such that $\delta(I) \subseteq R$. Since $Iq \subseteq R$, it also follows that $qI \subseteq R$. Thus $Iq + qI \subseteq R$ and we see that $q \in Q$, as desired. □

The smash product $Q\#u(L)$ is often very useful in examining the action of L on R. The next proposition, which holds for all prime rings, shows that $Q\#u(L)$ satisfies an important bimodule intersection property provided $L_{inn} = 0$ and some additional technical conditions are satisfied.

Proposition 9. *Let R be a prime ring with extended center C acted on by the finite dimensional restricted C-Lie superalgebra L. In addition, suppose that the action of L is C-linear and the action of σ is both X-outer and C-linear. If $L_{inn} = 0$, then every nonzero (R, R)-bimodule of $Q\#u(L)$ intersects R nontrivially.*

Proof. Let $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$ denote, respectively, a C-basis for L_0 and a C-basis for L_1. Every element of $u(L)$ is spanned over C by Poincaré-Birkhoff-Witt monomials of the form

$${x_1}^{i_1} \cdots {x_n}^{i_n} {y_1}^{j_1} \cdots {y_m}^{j_m},$$

where the exponent of each x_i is at most $p - 1$ and the exponent of each y_j is at most 1. We define the degree of

$${x_1}^{i_1} \cdots {x_n}^{i_n} {y_1}^{j_1} \cdots {y_m}^{j_m}$$

to be

$$i_1 + \cdots i_n + j_1 + \cdots j_m.$$

Observe that every element of $Q\#u(L)$ can be written uniquely as $\sum q_\Delta \Delta$, where $q_\Delta \in Q$ and the sum is taken over all Poincaré-Birkhoff-Witt monomials in $u(L)$. The support of $\sum q_\Delta \Delta$ is defined to be those monomials Δ such that $q_\Delta \neq 0$. The degree of $\sum q_\Delta \Delta$ is then defined to be the largest degree of any monomial in its support.

Let B be a nonzero (R, R)-bimodule of $Q\#u(L)$ and let N be the smallest degree of a nonzero element in B. Since every nonzero (R, R)-bimodule of Q certainly intersects R nontrivially, it suffices to show that $N = 0$. Therefore, by way of contradiction, we will assume that $N \geq 1$. From among all elements of B of degree N, let w be one with the minimal number of monomials of degree N in its support. Let

$$\eta = {x_1}^{i_1} \cdots {x_n}^{i_n} {y_1}^{j_1} \cdots {y_m}^{j_m}$$

be a monomial of degree $N - 1$ which was obtained by taking a degree N monomial in the support of w and lowering one of the exponents by 1. Next, let $\{\Delta_1, \ldots, \Delta_t\}$ be the monomials in the support of w from which we could obtain η by decreasing one of the exponents by 1. Let I be those elements of R which appear as a coefficient of Δ_1 in an element of B of degree N which has the same degree N monomials as w in its support. Observe that since B is an (R, R)-bimodule, we know that $I \cup \{0\}$ is a nonzero ideal of R. Since R is prime, we may assume that I contains some $a \neq 0$ such that $\sigma(a) = a$. Therefore, without loss of generality, we may assume that a is the coefficient of Δ_1 in w.

If Δ is a Poincaré-Birkhoff-Witt monomial in $u(L)$, let M_Δ denote the sum of the exponents of $y_1, \dots, y_m$. As a shorthand, let $M = M_{\Delta_1}$. It is clear, for every $\Delta_i \in \{\Delta_1, \dots, \Delta_t\}$, that either M_{Δ_i} is equal to M or differs from M by 1. Recall that in $Q\#u(L)$, if $s \in Q$ and d is the derivation corresponding to $x \in L_0$, then

$$xs = sx + d(s).$$

Similarly, if $s \in Q$ and δ is the σ-derivation corresponding to $y \in L_1$, then

$$ys = \sigma(s)y + \delta(s).$$

Now, if $r \in R$, let

$$v = arw - w\sigma^M(ra).$$

In light of the multiplication in $Q\#u(L)$, we see that the coefficient of Δ_1 in v is

$$ara - a\sigma^M(\sigma^M(ra)) = ara - ara = 0.$$

Thus v is an element of B of degree at most N having fewer degree N monomials in its support than w. Therefore, by the minimality of w, we have $v = 0$.

For $k > 1$, let a_k denote the coefficient of Δ_k in w. Suppose that, for some Δ_k, we have $M \neq M_{\Delta_k}$. Since M and M_{Δ_k} differ by 1, the coefficient of Δ_k in v is

$$0 = ara_k - a_k\sigma(r)a.$$

Thus, for all $r \in R$, $ara_k = a_k\sigma(r)a$. Since $a_k \neq 0$, this contradicts the fact that σ is X-outer. Therefore $M = M_{\Delta_k}$, for every Δ_k.

As a result, for $k > 1$, the coefficient of Δ_k in v is $0 = ara_k - a_kra$. Thus $ara_k = a_kra$, for all $r \in R$, which implies that there exists some $\lambda_k \in C$ such that $a_k = \lambda_k a$. If we let b denote the coefficient of η in w, then our argument now splits into the cases where $M = M_\eta$ and $M \neq M_\eta$.

If $M = M_\eta$, let d_k be the derivation corresponding to the element of L_0 where the exponent in Δ_k exceeds the exponent in η. If we let $r \in R$, then

$$\begin{aligned} ara_k\Delta_k - a_k\Delta_k\sigma^M(ra) = &-i_ka_kd_k(ra)\eta \\ &+ \text{ other terms of degree at most } N-1. \end{aligned}$$

This implies that the coefficient of η in v is

$$arb - bra - \sum \lambda_k i_k a d_k(ra).$$

Since $v = 0$, this implies that

$$\sum i_k\lambda_k a d_k(ra) = arb - bra.$$

By the result of Bergen and Montgomery [BM], this implies that the derivation $\sum i_k\lambda_k d_k$ is X-inner. Since $\lambda_1 = 1$ and i_i is invertible in C, we have obtained the contradiction that $\sum i_k\lambda_k d_k$ is a nonzero element of L_{inn}.

Thus we are now in the case where M and M_η differ by 1. If we let $r \in R$, then

$$\begin{aligned} ara_k\Delta_k - a_k\Delta_k\sigma^M(ra) = &-a_k\delta_k(ra)\eta \\ &+ \text{ other terms of degree at most } N-1. \end{aligned}$$

This implies that the coefficient of η in v is

$$arb - b\sigma(ra) - \sum \lambda_k a\delta_k(ra).$$

Since $v = 0$, this implies that

$$\sum \lambda_k a\delta(ra) = arb - b\sigma(r)a.$$

Proposition 8 implies that the σ-derivation $\sum \lambda_k \delta_k$ is X-inner. Since $\lambda_1 = 1$, we have obtained the contradiction that $\sum \lambda_k \delta_k$ is a nonzero element of L_{inn}, thereby concluding the proof. □

Our main result will deal with graded-reduced rings. Such rings are always semiprime but need not be prime. Therefore we need to extend Proposition 9 from prime rings to semiprime rings.

Corollary 10. *Let R be a semiprime ring with extended center C acted on by the finitely generated restricted C-Lie superalgebra L. In addition, suppose that the action of L is C-linear and the action of σ is both X-outer and C-linear. If $L_{inn} = 0$, then every nonzero (R, R)-bimodule of $Q\#u(L)$ intersects R nontrivially.*

Proof. Let B be a nonzero (R, R)-bimodule of $Q\#u(L)$ and let M be a maximal ideal of C. Since the actions of σ and L are C-linear, there are induced actions of σ and L on $\bar{Q}$. We can let $\bar{\sigma}$ denote the automorphism and $\bar{L}$ the $\bar{C}$-Lie superalgebra corresponding to these induced actions on $\bar{Q}$. Now choose M so that the image $\bar{B}$ of B in $\bar{Q}\#u(\bar{L})$ is nonzero.

We first need to show that the action of $\bar{\sigma}$ is X-outer. By way of contradiction, suppose there exists some $a \in Q$ such that $\bar{a}$ is nonzero in $\bar{Q}$ and

$$\bar{a}\bar{r} = \bar{\sigma}(\bar{r})\bar{a},$$

for all $r \in R$. This means that there exists some $c \in C \backslash M$ such that

$$(ca)r = \sigma(r)(ca),$$

for all $r \in R$. Since $\bar{c}$ is invertible in $\bar{C}$, $\overline{ca}$ is nonzero in $\bar{Q}$, hence ca is nonzero in Q. But this contradicts the fact that σ is X-outer on R. Thus $\bar{\sigma}$ is X-outer on $\bar{R}$.

Next, we claim that $\bar{L}_{inn} = 0$. If not, then there exist $q \in Q$, $c_i \in C$, and $\delta_i \in L$ such that every $\bar{c}_i$ is nonzero and $\sum \bar{c}_i \bar{\delta}_i$ is the derivation or $\bar{\sigma}$-derivation of $\bar{Q}$ induced by $\bar{q}$. This means that there exists some $c \in C \backslash M$ such that $\sum cc_i \delta_i$ is the derivation or σ-derivation of Q induced by cq. Since $L_{inn} = 0$, this implies that $cc_i = 0$, for all i, but this contradicts the fact that each $\bar{c}_i$ is nonzero.

Having shown that $\bar{\sigma}$ is X-outer and $\bar{L}_{inn} = 0$, we can apply Proposition 9 to assert that $\bar{B} \cap \bar{Q} \neq 0$. Therefore there exists some $0 \neq \bar{f} \in \bar{B} \cap \bar{Q}$ such than f is of the form

$$f = a_0 + \sum a_\Delta \Delta \in Q\#u(L),$$

where Δ runs through the Poincaré-Birkhoff-Witt monomials of $u(L)$ other than 1 and $a_0, a_\Delta \in Q$. Since $\bar{f} \in \bar{Q}$, there exists some $c \in C \backslash M$ such that $ca_0 \neq 0$ and $ca_\Delta = 0$, for all Δ. Thus cf is a nonzero element of $B \cap Q$ and so, B intersects Q nontrivially. □

Finite dimensional Hopf algebras H are Frobenius algebras and therefore contain nonzero elements $\int$, known as left integrals, such that $\omega(H)\int = 0$, where $\omega(H)$ is the augmentation ideal of H. It then follows that whenever H acts on a ring R, then the action of $\int$ sends R into R^H. Thus the existence of left integrals is very useful in producing invariants. In our situation, since C need not be a field, we are not automatically guaranteed that $u(L)$ contains a nonzero element t such that $Lt = 0$. However, the next lemma shows that such an element does indeed exist in our case.

Lemma 11. *Let R be a semiprime ring with extended center C acted on by the finitely generated restricted C-Lie superalgebra L. If the action of L is C-linear, then there exists a nonzero element $t \in u(L)$ such that $Lt = 0$.*

Proof. Let $x_1, \ldots, x_n$ generate L as a C-module and let M be a maximal ideal of C. As in the proof of Lemma 10, every element of L also acts on $\bar{Q}$ and we can let $\bar{L}$ denote the Lie superalgebra over the field $\bar{C}$ induced by the action of L on $\bar{Q}$. Observe that if we localize $u(L)$ at M to obtain $\overline{u(L)}$, then $\overline{u(L)} = u(\bar{L})$. Since $u(\bar{L})$ is a Frobenius algebra, so is $\overline{u(L)}$. As a result, $\overline{u(L)}$ contains a nonzero element $\int$ such that $\bar{x_i} \int = 0$, for all i.

Note that there exists $c \in C \backslash M$ and $w \in u(L)$ such that $c \int$ is the image in $\overline{u(L)}$ of w. Therefore, for each i, there exists some $c_i \in C \backslash M$ such that $c_i x_i w = 0$. Finally, let

$$t = (c_1 \cdots c_n) w.$$

Then t is a nonzero element of $u(L)$ such that $x_i t = 0$, for all i. Hence $Lt = 0$, as desired. □

Our next lemma is a slight generalization of [BG, Lemma 3.2]. It shows that, with the insertion of various idempotents, modules over commutative von Neumann regular rings behave very much like vector spaces over fields.

Lemma 12. *Let $M = Cx_1 + \cdots + Cx_n$ be a finitely generated C-module and N a proper submodule, where C is a commutative von Neumann regular ring. Then there exists a nonzero idempotent $e \in C$ and a re-ordering of the x_i such that $eM \neq 0$ and*

$$eM = (Cex_1 + \cdots + Cex_m) \oplus eN,$$

where $m < n$ and eN is generated over C by at most $n - m$ elements.

Proof. Suppose e_1 is a nonzero idempotent of C such that $e_1 M \neq 0$ and

$$e_1 M = (Ce_1 x_1 + \cdots + Ce_1 x_k) + e_1 N,$$

where $k \leq n$. If

$$(Ce_1 x_1 + \cdots + Ce_1 x_k) \cap e_1 N \neq 0,$$

let $w = c_1 e_1 x_1 + \cdots + c_k e_1 x_k$ belong to this intersection, where each $c_i \in C$. By re-ordering the set $\{x_1, \ldots, x_k\}$, we may assume that $c_k e_1 x_k \neq 0$. Since C is von Neumann regular, there exists $c \in C$ such that ${c_k}^2 c = c_k$. Therefore $f = cc_k$ is a nonzero idempotent such that

$$fe_1 x_k = cw - cc_1 e_1 x_1 - \cdots - cc_{k-1} e_1 x_{k-1}.$$

Observe that $fe_1 x_k \neq 0$, since $c_k f e_1 x_k = c{c_k}^2 e_1 x_k = c_k e_1 x_k \neq 0$. As a result,

$$0 \neq fe_1 M = Cfe_1 x_1 + \cdots + Cfe_1 x_{k-1} + fe_1 N.$$

This says that if $(Ce_1 x_1 + \cdots Ce_1 x_k) \cap e_1 N \neq 0$, then there exists an idempotent $e_2 = fe_1$ such that we can re-order the elements of $\{e_1 x_1, \ldots, e_1 x_k\}$ and then replace it by the smaller set $\{e_2 x_1, \ldots, e_2 x_{k-1}\}$.

Starting with the idempotent 1 and the set $\{x_1, \ldots, x_n\}$, it is certainly true that

$$1M \neq 0, \quad 1M = Cx_1 + \cdots + Cx_n + 1N, \quad \text{and} \quad (Cx_1 + \cdots + Cx_n) \cap 1N \neq 0.$$

Therefore we can repeatedly apply the above procedure to produce a nonzero idempotent $e \in C$ such that $eM \neq 0$ and

$$eM = (Cex_1 + \cdots + Cex_m) \oplus eN,$$

where $m < n$.

Finally, since $eN \simeq eM/(Cex_1 + \cdots + Cex_m)$, it follows that eN is generated over C by at most $n - m$ elements. □

We now have all the pieces needed to prove the main result of this paper.

Theorem 13. *If R is a graded-reduced ring of characteristic $p > 2$ acted on by a finitely generated restricted K-Lie superalgebra L, where $K \subseteq C_0$, then $R^L \neq 0$.*

Proof. We will proceed by induction by showing that, for every positive integer n, whenever a graded-reduced ring R is acted on by a restricted K-Lie superalgebra L such that

$$K \subseteq C_0, \quad L_0 = Kx_1 + \cdots + Kx_s, \quad L_1 = Ky_1 + \cdots + Ky_t, \quad \text{and} \quad n = s + t,$$

then it must be the case that $R^L \neq 0$. We begin with the case where $n = 1$; therefore either $s = 1$ or $t = 1$. If $s = 1$, then L is a Lie algebra and the result follows by applying either the result of Beidar and Grzeszczuk [BG] or more general results on the actions of a single algebraic derivation. On the other hand, if $t = 1$ and if δ is the σ-derivation corresponding to the action of y_1, then $\delta^2 = 0$. Thus δ clearly has nonzero invariants and so, $R^L \neq 0$. Having shown that $R^L \neq 0$ in the $n = 1$ case, we may now assume that $n > 1$.

By Lemma 7, if the action of either L_1 and σ fails to be C-linear, then $R^L \neq 0$. Therefore, for the remainder of the proof, we may assume that the actions of both L_1 and σ are C-linear. Since we are now in the case where $C = C_0$, it follows that $Der_\sigma(R, Q)$ is a module over C. We can let CL, CL_0, CL_1 denote the C-submodules of $Der_\sigma(R, Q)$ generated by L, L_0, L_1, respectively. Although the action of L_1 is C-linear, it is possible that the action of L_0 is not C-linear. Although CL_0 is a C-module which is both a Lie ring and closed under taking pth powers, it is not technically a Lie algebra over C. In the work of Kharchenko [K3] and Beidar-Grzeszczuk [BG], these objects are referred to as restricted Lie ∂-algebras. In [BG], it is shown that whenever a restricted Lie ∂-algebra L_0 is finitely generated over C and acts on a reduced ring R, then $R^{L_0} \neq 0$.

Let U be the restricted subalgebra of CL generated over C by L_1. Since

$$[L_1, L_1] \oplus L_1$$

is an ideal of L, it is clear that U is an ideal of the Lie superring CL. In addition, since every element of L_1 is C-linear, U is a Lie superalgebra over C. By Lemma 12, there exists some $e = e^2 \in C$ such that $eU \neq 0$ and either $eCL = eU$ or eU is generated over C by fewer than n elements.

In the latter case, let J be a σ-stable essential ideal of R such that $Je \subseteq R$. Lemma 1(3) tells us that eU is a restricted Lie superalgebra over Ce acting on Je. To show that $R^L \neq 0$, it suffices to show that $(Je)^{eL} \neq 0$. Therefore, without loss of generality, we may reduce down the case where U is generated over C by less that n elements. Now, let $M \neq 0$ be a σ-stable ideal of R. By Lemma 1(2), there exists some $f = f^2 \in C$ such that $fU \neq 0$ and fU acts on M so that we can identify this action with the action of U on M. However, since fU is generated over Cf by less than n generators, the induction hypothesis implies that fU acts on M with nonzero invariants. Thus

$$Q^U \cap M = M^U \neq 0.$$

Furthermore, since U is an ideal of CL, it follows that Q^U is L-stable. By Lemma 2(2), L acts on R^U and the Lie algebra L_0 acts on $(R^U)_0$. By the result of Beidar-Grzeszczuk [BG], $((R^U)_0)^{L_0} \neq 0$. However,

$$((R^U)_0)^{L_0} \subseteq R^L.$$

Thus, in this case, $R^L \neq 0$.

The remaining case is that $eCL = eU$. By Lemma 1(3), there exists a σ-stable essential ideal J of R such that eU acts on Je. Since it now suffices to show that $(Je)^{eU} \neq 0$, without loss of generality, we can assume that $L = U$. As a result, all elements of L are C-linear and we can consider L to be a Lie superalgebra over C. Therefore, without loss of generality, may assume that $L = CL, L_0 = CL_0$, and $L_1 = CL_1$. If the sum

$$L = Cx_1 + \cdots + Cx_s + Cy_1 + \cdots + Cy_t,$$

where $n = s + t$, is not direct that there is an idempotent $e \in C$ such that eL is generated over Ce by less than n generators. In this case, nonzero invariants would exist by applying the induction hypothesis to the action of eL on a suitable nonzero ideal of R. Therefore we may reduce to the case where

$$L = Cx_1 \oplus \cdots \oplus Cx_s \oplus Cy_1 \oplus \cdots \oplus Cy_t.$$

The remainder of the proof now splits into cases where either $L_{inn} = 0$ or $L_{inn} \neq 0$.

For the moment, let us assume that $L_{inn} = 0$. Since the action of both L and σ are C-linear, we can apply Lemma 11 to assert that there exists some $t \neq 0$ such that $Lt = 0$ in $u(L)$. Since $Lt = 0$, it is clear that $t(R) \subseteq Q^L$. Let J be an essential ideal of R such that $t(J) \subseteq R$ and let

$$B = \{b \in Q\#u(L) \mid b(J) = 0\}.$$

Certainly B is an (R, R)-bimodule of $Q\#u(L)$. If $L_{inn} = 0$ and if $B \neq 0$, we can apply Corollary 10 to see that $B \cap R \neq 0$. However, if a is a nonzero element of $B \cap R$, the action of a on J is via left multiplication. But this leads to the contradiction $aJ = 0$. Thus $B = 0$ and so,

$$0 \neq t(J) \subseteq R^L.$$

Hence, in this case, $R^L \neq 0$.

For our final case, we may assume that $L_{inn} \neq 0$. By Lemma 12, there exists $e = e^2 \in C$ and a re-ordering of the x_i and y_j such that $eL_{inn} \neq 0$ and

$$eL = (Cex_1 \oplus \cdots \oplus Cex_{s'} \oplus Cey_1 \oplus \cdots \oplus Cey_{t'}) \oplus eL_{inn},$$

where $s' + t' < n$. Since it suffices to show that eL acts with nonzero invariants, Lemma 1(3) allows us to reduce to the case where

$$L = (Cx_1 \oplus \cdots \oplus Cx_{s'} \oplus Cy_1 \oplus \cdots \oplus Cy_{t'}) \oplus L_{inn},$$

where $s' + t' < n$.

In light of Lemma 4 and Lemma 2(2), $R^{L_{inn}} \neq 0$ and L acts on $R^{L_{inn}}$. But this implies that the quotient superalgebra L/L_{inn} acts on $R^{L_{inn}}$. However L/L_{inn} is generated over C by less than n elements, therefore the induction hypothesis implies that

$$(R^{L_{inn}})^{L/L_{inn}} \neq 0.$$

Since

$$R^L = (R^{L_{inn}})^{L/L_{inn}},$$

we have succeeded in showing that $R^L \neq 0$. □

The question which motivated this paper dealt with the existence of nonzero invariants when reduced rings were acted on by finite dimensional Hopf algebras. Recall that if $L_1 \neq 0$, then $u(L)$ is not a Hopf algebra but the smash product $H = u(L)\#G$ is a Hopf algebra which is neither commutative nor cocommutative. Using Theorem 13, it is now easy to prove that every nonzero H-stable subring of a reduced ring R acted on by $H = u(L)\#G$ contains nonzero invariants.

Corollary 14. *Let R be a reduced algebra over a field K of characteristic $p > 2$ acted on by a finite dimensional restricted K-Lie superalgebra L and let $H = u(L)\#G$, where G is the group of order 2 with the natural action on L. Then $A^H \neq 0$, for every nonzero H-stable subalgebra A of R*

Proof. First observe that since A is a subalgebra of R, the field K embeds in the even part of the extended center of A. Next, since A is H-stable, it follows that $\delta(A) \subseteq A$, for all $\delta \in L_0 \cup L_1$. Therefore L acts on A and we can apply Theorem 13 to assert that $A^L \neq 0$. Since σ commutes with the elements of L_0 and anti-commutes with the elements of L_1, it is clear A^L is G-stable. Since A is reduced, Kharchenko's result [K1] on group actions or a direct calculation shows that $(A^L)^G \neq 0$. However,

$$A^H = (A^L)^G,$$

thus $A^H \neq 0$, as desired. □

The existence of nonzero invariants provided by Corollary 14 can be combined with a recent result of Grzeszczuk and Hryniewicka [GH2, Theorem 4] to provide us with the final result of this paper.

Corollary 15. *Let R be a reduced algebra over a field K of characteristic $p > 2$ acted on by a finite dimensional restricted K-Lie superalgebra L and let $H = u(L)\#G$, where G is the group of order 2 with the natural action on L. If R^H satisfies a polynomial identity of degree d, then R satisfies a polynomial identity of degree dN, where N is the dimension of H.*

Proof. Theorem 4 of [GH2] examines the situation where reduced rings R are acted on by finite dimensional pointed Hopf algebras H such that every nonzero H-stable left ideal of R contains nonzero invariants. In this situation, they show that if R^H satisfies a polynomial identity of degree d, then R satisfies a polynomial identity of degree dN, where N is the dimension of H.

In light of Corollary 14 and the fact that $H = u(L)\#G$ is pointed, the result in [GH2] can immediately be applied to our situation, thereby proving the result. □

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