

JACOBSON RADICALS OF RING EXTENSIONS

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STRESZCZENIE. We extend existing results on the Jacobson radical of skew polynomial rings of derivation type when the base ring has no nonzero nil ideals. We then move to the more general situation of algebras with locally nilpotent skew derivations and examine the Jacobson radical of the algebra when the subalgebra of invariants has no nonzero nil ideals.

In [1], Amitsur showed that if R has no nonzero nil ideals, then the Jacobson radical of the polynomial ring $R[x]$ is 0. Subsequently, there has been a great deal of work examining the Jacobson radicals of more general ring extensions such as skew polynomial rings of automorphism type and of derivation type. For skew polynomial rings $R[x; \sigma]$ of automorphism type, it was shown in [2] that even if R is commutative and reduced, then $J(R[x; \sigma])$ can be nonzero. For skew polynomial rings $R[x; \delta]$ of derivation type, it is still unknown if $J(R[x; \delta])$ must be zero when R has no nonzero nil ideals.

Although the situation regarding $J(R[x; \delta])$ is still open, it was shown in [5] and [7] that if one assumes either that R is reduced or satisfies a polynomial identity or satisfies the ascending chain condition on right annihilators, then $J(R[x; \delta]) = 0$ whenever R has no nonzero nil ideals. The condition that R satisfies the ascending chain conditions on right annihilators of powers is weaker than R either being reduced or satisfying the ascending chain condition on annihilators. Also, for semiprime rings, the condition that every ideal of R contains a normalizing element is weaker than R satisfying a polynomial identity. Therefore our first two main results, which we now state, extend existing results on $J(R[x; \delta])$.

Theorem 2. *Let R be an algebra with no nonzero nil ideals satisfying the acc on right annihilators of powers.*

- (1) *If δ is a derivation of R , then $J(R[x; \delta]) = 0$.*
- (2) *If L is a Lie algebra acting on R as derivations, then $J(R\#U(L)) = 0$.*

Theorem 3. *Let R be a semiprime algebra where every nonzero ideal contains a normalizing element.*

- (1) *If δ is a derivation of R , then $J(R[x; \delta]) = 0$.*

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(2) If L is a Lie algebra acting on R as derivations, then $J(R\#U(L)) = 0$.

Next, suppose that B is an algebra with a q -skew derivation δ , where either q is not a root of 1 or R has characteristic 0 and $q = 1$. It is shown in [3] that the skew polynomial ring $B[x; \sigma, \delta]$ has a locally nilpotent q^{-1} -skew σ^{-1} -derivation d such that the B is the subalgebra of constants of d . Therefore, we can think of the relationship between an algebra and a skew polynomial extension as being a special case of the relationship between the subalgebra of constants of a locally nilpotent q -skew derivation and the original algebra.

Therefore it is natural to examine algebras R with a locally nilpotent q -skew σ -derivation d such that R^d has no nonzero nil ideals and then try examine when $J(R)$ is 0. If we look at the example in [2] on skew polynomial rings in the context of algebras R with locally nilpotent skew derivation d , it shows that $J(R)$ need not be 0, even if the constants of d are commutative and reduced. However, our next two main results, which we state below, illustrate there are many cases in which $J(R)$ is equal to 0.

Theorem 5. *Let R be an algebra with a locally nilpotent regular q -skew σ -derivation d , where either q is not a root of 1 or R has characteristic 0 and $q = 1$. If R^d is semiprime Goldie, then $J(R) = 0$.*

Theorem 10. *Let R be an algebra of characteristic 0 with a locally nilpotent regular σ -derivation d such that $d\sigma = \sigma d$. If R^d has no nonzero nil ideals, then $J(R) = 0$ in all of the following cases:*

- (1) d is a derivation and R^d satisfies the acc on right annihilators of powers,
- (2) σ has locally finite order, R is an algebra over an uncountable field, and R^d satisfies the acc on right annihilators of powers,
- (3) σ has locally finite order and R^d satisfies the acc on right annihilators,
- (4) σ has locally finite order and R^d is reduced,
- (5) σ has locally finite order and R^d satisfies a polynomial identity.

Observe that if $d = 0$, then $R^d = R$ and it is certainly possible that R^d has no nonzero nil ideals, yet $J(R)$ is not equal to 0. To avoid this type of situation, Theorems 5 and 10 both have the additional assumption that d is regular. This is a technical condition that is satisfied in many cases. In particular, the locally nilpotent q -skew derivation of $B[x; \sigma]$ having B as its constants is regular. If we remove the assumption that d is regular, then we can prove Theorems 6 and 11 in which we show that there are many cases in which $J(R^d)$ being 0 implies that $J(R) = 0$.

We will now introduce the terminology and notation that will be used throughout the paper. R will be an algebra with multiplicative identity over a field K . If σ is a K -linear automorphism of R , then a σ -derivation d is a K -linear map $d : R \rightarrow R$ such that

$$d(rs) = d(r)s + \sigma(r)d(s),$$

for all $r, s \in R$. The ring of constants R^d is defined as

$$R^d = \{r \in R \mid d(r) = 0\}.$$

A σ -derivation d is said to be *locally nilpotent* if for every $r \in R$, there exists $n = n(r) \geq 1$ such that $d^n(r) = 0$. If q is a nonzero element of K , we say that our σ -derivation is *q -skew* if

$$d\sigma(r) = q\sigma d(r),$$

for all $r \in R$. For $n \geq 1$, let

$$(n!)_q = \prod_{k=1}^n (1 + q + \cdots + q^{k-1}).$$

Then the q -binomial coefficient $\binom{n}{i}_q$ is defined as evaluation at $t = q$ of the polynomial function

$$\binom{n}{i}_t = \frac{(t^n - 1)(t^{n-1} - 1) \cdots (t^{n-i+1} - 1)}{(t^i - 1)(t^{i-1} - 1) \cdots (t - 1)}.$$

If q is not a root of unity, then

$$\binom{n}{k}_q = \frac{(n!)_q}{((n-k)!)_q (k!)_q}$$

is nonzero for all $n \geq k \geq 0$.

The following q -Leibniz Rule holds in a ring with q -skew σ -derivation d .

$$d^n(ab) = \sum_{j=0}^n \binom{n}{j}_q \sigma^{n-j} d^j(a) d^{n-j}(b)$$

for all $a, b \in R$ and $n \geq 0$.

For $m \geq 0$, let $R_m = \ker d^{m+1}$. Clearly, d is locally nilpotent if and only if $R = \bigcup_{m \geq 0} R_m$.

By the *degree* of an element $a \in R$, which we denote as $\deg(a)$, we mean the integer n such that $a \in R_n$ but $a \notin R_{n-1}$. The q -Leibniz Rule implies that $R_n R_m \subseteq R_{n+m}$, so R is a filtered algebra, with $R_0 = R^d$.

For subsets A, B of a ring R , we let $\text{r. ann}_A(B) = \{a \in A \mid Ba = 0\}$. We say that R satisfies the *acc on right annihilators of powers* if, for every $r \in R$, there exists $n \geq 1$ such that

$$\text{r. ann}_R(r) \subseteq \text{r. ann}_R(r^2) \subseteq \cdots \subseteq \text{r. ann}_R(r^n) = \text{r. ann}_R(r^{n+1}) = \cdots .$$

Observe that satisfying the acc on right annihilators of powers is a weaker condition than either satisfying the acc on right annihilators or being reduced. When d is a locally nilpotent σ -derivation of a ring R , we say that d is *right regular* (or simply *regular*) if $\text{r. ann}_{R^d}(d(R) \cap R^d) = 0$. Observe that d being regular is equivalent to $\text{r. ann}_R(d(R) \cap R^d) = 0$.

If $0 \neq r \in R$, we say that r is *normalizing* if $rR = Rr$. Semiprime rings satisfying a polynomial identity have the property that every nonzero ideal contains a nonzero central element. Observe that, for semiprime rings R , the condition that every nonzero ideal contains a normalizing element is weaker than the condition that R satisfies a polynomial identity.

We say that an automorphism σ has *locally finite order* if, for every $r \in R$, there exists $n \geq 1$ such that $\sigma^n(r) = r$. Observe that if $d \neq 0$ is a q -skew σ -derivation and if σ has locally finite order, then q must be a root of 1.

We begin our work with

Lemma 1. *Let R be an algebra with derivation δ . If $a \in J(R[x; \delta]) \cap R$ such that $\text{r. ann}_R(a) = \text{r. ann}_R(a^2)$, then $a = 0$.*

Dowód. Since $a \in J(R[x; \delta])$, it follows that $xa \in J(R[x; \delta])$. Therefore xa has a quasi-inverse $b(x) = b_n x^n + \cdots + b_1 x + b_0 \in R[x; \delta]$ and we have

$$(1) \quad xa + (b_n x^n + \cdots + b_1 x + b_0) = xa(b_n x^n + \cdots + b_1 x + b_0).$$

If $ab(x) = 0$, then multiplying (1) on the left by a gives us $axa = 0$. Since $axa = a^2 x + a\delta(a)$, we see that $a^2 = 0$. As a result, $\text{r. ann}_R(a) = \text{r. ann}_R(a^2) = R$, which immediately implies that $a = 0$.

Now suppose that $ab(x) \neq 0$; therefore there is a largest integer $m \geq 0$ such that $ab_m \neq 0$. Multiplying equation (1) on the left by a now gives us

$$(2) \quad axa + (ab_m x^m + \cdots + ab_1 x + ab_0) = axa(b_m x^m + \cdots + b_1 x + b_0).$$

Since $b_m \notin \text{r. ann}_R(a)$, it follows that $b_m \notin \text{r. ann}_R(a^2)$ and $a^2 b_m \neq 0$. Therefore the right hand side of equation (2) has degree $m + 1$ as the coefficient of x^{m+1} is $a^2 b_m$. If $m \geq 1$, then the degree of the right hand side of equation (2) exceeds the degree of the left hand side, a contradiction. Thus $m = 0$ and equation (2) now becomes

$$(3) \quad axa + ab_0 = axab_0.$$

If we look at the coefficient of x on each side of equation (3), we see that $a^2 = a^2 b_0$. Therefore $1 - b_0 \in \text{r. ann}_R(a^2) = \text{r. ann}_R(a)$, hence $a = ab_0$. At this point, equation (3) simplifies to

$$axa + a = axa,$$

hence $a = 0$, contradicting the assumption that $ab(x) \neq 0$. Thus $a = 0$, as required. \square

The construction of $R[x; \delta]$ using a single derivation can be extended to construct the smash product $R\#U(L)$, where L is a Lie algebra acting on R as derivations and $U(L)$ is the universal enveloping algebra of L . For more details on $R\#U(L)$, we refer the reader to [5]. We can now prove our first main result on the Jacobson radical.

Theorem 2. *Let R be an algebra with no nonzero nil ideals satisfying the acc on right annihilators of powers.*

(1) *If δ is a derivation of R , then $J(R[x; \delta]) = 0$.*

(2) *If L is a Lie algebra acting on R as derivations, then $J(R\#U(L)) = 0$.*

Dowód. By way of contradiction, in order to prove part (1), we will assume that $J(R[x; \delta]) \neq 0$. Since R has no nonzero nil ideals, a special case of Proposition 3.7 of [5] asserts $J(R[x; \delta]) \cap R \neq 0$. If $\alpha \in J(R[x; \delta]) \cap R$, then the acc condition on right annihilators of powers implies that there exists $n \geq 1$ such that $\text{r. ann}_R(\alpha^n) = \text{r. ann}_R(\alpha^{n+1})$. If we let $a = \alpha^n$, it follows that $\text{r. ann}_R(a) = \text{r. ann}_R(a^2)$. However, Lemma 1 tells us that $a = 0$, hence $\alpha^n = 0$. Therefore every element of $J(R[x; \delta]) \cap R$ is nilpotent, contradicting the assumption that R has no nonzero nil ideals. Thus $J(R[x; \delta]) = 0$, proving part (1).

For part (2), by way of contradiction, we will assume that $J(R\#U(L)) \neq 0$. Since R has no nonzero nil ideals, Proposition 3.7 of [5] asserts that $J(R\#U(L)) \cap R \neq 0$. Next, let $0 \neq x \in L$ and let δ be the derivation of R corresponding to x . A well known consequence of the Poincaré-Birkhoff-Witt theorem, which can be found in Lemma 3.8 of [5], tells us that

$$J(R\#U(L)) \cap R[x; \delta] \subseteq J(R[x; \delta]).$$

However, part (1) showed that $J(R[x; \delta]) = 0$. Therefore, we now have

$$0 \neq J(R\#U(L)) \cap R \subseteq J(R\#U(L)) \cap R[x; \delta] \subseteq J(R[x; \delta]) = 0,$$

a contradiction. Thus $J(R\#U(L)) = 0$, proving part (2). \square

We can also use Lemma 1 to prove our second main result.

Theorem 3. *Let R be a semiprime algebra where every nonzero ideal contains a normalizing element.*

- (1) *If δ is a derivation of R , then $J(R[x; \delta]) = 0$.*
- (2) *If L is a Lie algebra acting on R as derivations, then $J(R\#U(L)) = 0$.*

Dowód. The proof of part (2) will follow from part (1) in the identical manner as in Theorem 2. Therefore, it will suffice to prove part (1). As in the proof of Theorem 2, by way of contradiction, we will assume that $J(R[x; \delta]) \neq 0$ and it again follows that $J(R[x; \delta]) \cap R \neq 0$. Therefore there exists a normalizing element $a \in J(R[x; \delta]) \cap R$. Since $aR = Ra$, observe that if $b \in \text{r. ann}_R(a^2)$ we have

$$(Rab)^2 = (Rab)(Rab) \subseteq R(aR)ab = R(Ra)ab = Ra^2b = 0.$$

Therefore Rab is a nilpotent left ideal of R , which implies that $Rab = 0$. As a result, $ab = 0$ and $b \in \text{r. ann}_R(a)$. Having shown that $\text{r. ann}_R(a) = \text{r. ann}_R(a^2)$, we can apply Lemma 1 to conclude that $a = 0$. However this contradicts that a is normalizing, thus it is the case $J(R[x; \delta]) = 0$, proving (1). \square

For the remainder of this paper, we will examine algebras R with a locally nilpotent q -skew σ -derivation d . We will focus on conditions on R^d that will guarantee that $J(R) = 0$.

Lemma 4. *Let R be an algebra with a locally nilpotent q -skew σ -derivation d , where either q is not a root of 1 or R has characteristic 0 and $q = 1$. If R^d is semiprime Goldie and $J(R) \neq 0$, then $J(R) \cap R^d \neq 0$.*

Dowód. Let n denote the smallest degree of a nonzero element of $J(R)$. If $n = 0$, there is nothing to prove. Therefore, by way of contradiction, we will assume that $n > 0$. Next, let A be the elements of $J(R)$ of degree n and consider the set $d^n(A) \cup \{0\}$. Since d is q -skew, we know that $\sigma^i(R^d) = R^d$, for all $i \in \mathbb{Z}$. If $\alpha \in R^d$ and $a \in A$, then $\sigma^{-n}(\alpha)a, a\alpha \in A$, which implies that

$$(4) \quad \alpha d^n(a) = d^n(\sigma^{-n}(\alpha)a) \in d^n(A) \cup \{0\} \quad \text{and} \quad d^n(a)\alpha = d^n(a\alpha) \in d^n(A) \cup \{0\}.$$

Thus $d^n(A) \cup \{0\}$ is an ideal of R^d .

We now let $C = \text{r. ann}_{R^d}(d^n(A) \cup \{0\})$ and $B = \text{r. ann}_{R^d}(C)$. Since B is the annihilator of an ideal in R^d and $d^n(A) \cup \{0\}$ is an essential ideal of B , it follows that there exists some $a \in A$ such that $d^n(a)$ is regular in B . The set C is σ -stable and is certainly both the left and right annihilator of $d^n(A)$ in R^d . Therefore if we now suppose that the element α from equation (4) also belongs to C , then equation (4) becomes

$$d^n(\sigma^{-n}(\alpha)a) = \alpha d^n(a) \in Cd^n(A) = 0 \quad \text{and} \quad d^n(a)\alpha = d^n(a)\alpha \in d^n(A)C = 0.$$

Thus $\sigma^{-n}(\alpha)a, a\alpha$ are elements of $J(R)$ with degrees less than n , hence they must both be 0. This tells us that $Ca = aC = 0$.

Since $a \in J(R)$, it has a quasi-inverse $r \in R$ and we have

$$(5) \quad a + r = ar = ra.$$

If $\alpha \in C$, then multiplying this equation on the right by α gives us $r\alpha = 0$, whereas multiplying it on the left by α gives us $\alpha r = 0$. Thus $Cr = rC = 0$. Furthermore, if we let m denote the degree of d on r , then $d^m(r) \in R^d$ and

$$0 = d^m(rC) = d^m(r)C.$$

Since B is both the left and right annihilator of C in R^d , the equation above shows that $d^m(r) \in B$. If $m > 0$, then $n + m$ exceeds both n and m and applying d^{n+m} to equation (5) gives us

$$0 = d^{n+m}(a + r) = d^{n+m}(ar) = \binom{n+m}{n}_q \sigma^m(d^n(a))d^m(r)$$

However, this is a contradiction as $\sigma^m(d^n(a))$ is a regular element of B , $d^m(r)$ is a nonzero element of B , and $\binom{n+m}{n}_q$ is a nonzero element of the base field. In light of this, it must be the case that $m = 0$. However, if we now apply d to equation (5), we obtain

$$d(a) = d(a)r.$$

If we multiply equation (5) on the left by $d(a)$, it now simplifies down to

$$d(a)a + d(a) = d(a)a.$$

This immediately implies that $d(a) = 0$, which contradicts that a has degree $n > 0$, concluding the proof. \square

We can now prove our third main result.

Theorem 5. *Let R be an algebra with a locally nilpotent regular q -skew σ -derivation d , where either q is not a root of 1 or R has characteristic 0 and $q = 1$. If R^d is semiprime Goldie, then $J(R) = 0$.*

Dowód. Suppose, by way of contradiction, that $J(R) \neq 0$. Then, by Lemma 4, $J(R) \cap R^d \neq 0$. Let $C = \text{r. ann}_{R^d}(J(R) \cap R^d)$ and $B = \text{r. ann}_{R^d}(C)$. Since d is regular, $d(R) \cap R^d$ is an essential ideal of R^d . Combined with the fact that $J(R) \cap R^d$ is an essential ideal of B , we see that

$$(J(R) \cap R^d) \cap (d(R) \cap R^d) = J(R) \cap R^d \cap d(R)$$

is an essential ideal of B . Therefore there exists some $a \in J(R) \cap R^d \cap d(R)$ such that a is regular in B and $a = d(x)$, for some $x \in R$. Since $xa \in J(R)$, it has a quasi-inverse $r \in R$ and we have

$$(6) \quad xa + r = rxa = xar$$

If we multiply equation (6) on the right by C , we see that $rC = 0$. If m is the degree of r and we apply d^m to rC , we obtain $0 = d^m(rC) = d^m(r)C$. Since C has the same left and right annihilators in R^d and $d^m(r) \in R^d$, it follows that $d^m(r)$ is a nonzero element of B . If $m > 0$, then $d^{m+1}(xa) = d^{m+1}(r) = 0$ and applying d^{m+1} to equation (6) gives us

$$0 = d^{m+1}(xar) = \binom{m+1}{1}_q \sigma^m(d(xa))d^m(r) = \binom{m+1}{1}_q a^2 d^m(r).$$

However, this is a contradiction as a^2 is a regular element of B , $d^m(r)$ is a nonzero element of B , and $\binom{m+1}{1}_q$ is a nonzero element of the base field. Therefore, it must be that $m = 0$. If we now apply d to equation (6), we obtain

$$a^2 = a^2r.$$

This implies that $a^2(1 - r) = 0$, which is a contradiction as $1 - r$ is invertible in R and a is not nilpotent. Thus $J(R) = 0$. \square

If we remove the condition that d is regular, we can adapt Theorem 5 to the situation where $J(R^d) = 0$.

Theorem 6. *Let R be an algebra with a locally nilpotent q -skew σ -derivation d , where either q is not a root of 1 or R has characteristic 0 and $q = 1$. If R^d is Goldie with $J(R^d) = 0$, then $J(R) = 0$.*

Dowód. By Proposition 1 of [4], there exist ideals A and B of R which are d -stable and σ -stable such that $B \subseteq R^d$, $A = \text{r. ann}_R(B)$, $\text{r. ann}_R(A \oplus B) = 0$, and $\text{r. ann}_A(d(A) \cap A^d) = 0$. Observe that $A^d = \text{r. ann}_{R^d}(B)$, thus A^d is the annihilator of an ideal of R^d and is therefore also a semiprime Goldie ring. However, the condition that $\text{r. ann}_A(d(A) \cap A^d) = 0$ is equivalent to d being regular when restricted to A . Therefore we can apply Theorem 5 to A to conclude that $J(A) = 0$.

Next, if $J(B) \neq 0$, then $BJ(B)B$ is a nonzero quasi-regular ideal of R^d , contradicting that $J(R^d) = 0$. Therefore it is also the case that $J(B) = 0$. Since $J(A) = J(B) = 0$, we also know that $J(A \oplus B) = J(A) \oplus J(B) = 0$. Finally, since $A \oplus B$ is an essential ideal of R , if $J(R) \neq 0$, then $(A \oplus B) \cap J(R)$ is a nonzero quasi-regular ideal of R contained in $A \oplus B$. However this contradicts the fact that $J(A \oplus B) = 0$, proving the result. \square

For the remainder of this paper, we will restrict our work to algebras in characteristic 0.

Lemma 7. *Let S be an algebra of characteristic 0 (not necessarily with 1) with no nil ideals and an automorphism σ of locally finite order. Then S^σ is not nil if any of the following conditions hold:*

- (1) S is reduced,
- (2) S is an algebra over an uncountable field,
- (3) S satisfies the acc on right annihilators,
- (4) S satisfies a polynomial identity.

Dowód. Let $b \in S$ such that b is not nilpotent and then let $n \geq 1$ be such that $\sigma^n(b) = b$. Observe that σ has finite order when acting on S^{σ^n} and the fixed ring of this action is also S^σ . Since S^{σ^n} is not nilpotent, the Bergman-Isaacs theorem [6] asserts that S^σ is also not nilpotent. Note that handles case (1) for if $S^\sigma \neq 0$, then it is also not nil.

For case (2), by way of contradiction, suppose S^σ is nil. The previous paragraph asserts that there exists some nonzero $a \in S^\sigma$. Now let $t \in SaS$ and let $m \geq 1$ be such that $\sigma^m(t) = t$. The automorphism σ now acts with finite order on S^{σ^m} with fixed ring S^σ . If the ground field is uncountable, then S^σ being nil implies that S^{σ^m} is also nil. Thus t is nilpotent, hence SaS is a nil ideal of S , a contradiction. This completes case (2).

For case (3), since S satisfies the acc on right annihilators, so does S^σ . In this situation, S^σ being nil implies that it contains a nonzero nilpotent ideal I . If a is a nonzero element of I ,

let $t \in SaS$. Therefore, there exist $r_i, s_i \in S$ such that $t = \sum_i r_i a s_i$ and there exists let $m \geq 1$ such that each r_i and s_i is fixed by σ^m . Therefore $t \in S^{\sigma^m} a S^{\sigma^m}$. Another application of the Bergman-Isaacs theorem is that

$$P(S^\sigma) = P(S^{\sigma^m}) \cap S^\sigma,$$

where $P(S^\sigma)$ and $P(S^{\sigma^m})$ are, respectively, the prime radicals of S^σ and S^{σ^m} . Since $a \in I \subseteq P(S^\sigma)$, it follows that $a \in P(S^{\sigma^m})$, which implies that $t \in S^{\sigma^m} a S^{\sigma^m} \subseteq P(S^{\sigma^m})$. However, the prime radical of a ring is always nil, hence t is nilpotent. Thus SaS is a nil ideal of S , a contradiction. This completes case (3).

The proof of case (3) was based on S^σ containing a nonzero nilpotent ideal. However, if S^σ is nil and satisfies a polynomial identity, then S^σ again contains a nonzero nilpotent ideal I . From this point on, the proof of case (4) is the same as the proof of case (3). \square

Although in a different setting, the next lemma is somewhat similar to Lemma 4.

Lemma 8. *Let R be an algebra of characteristic 0 with a locally nilpotent σ -derivation d such that $d\sigma = \sigma d$. If R^d has no nonzero nil ideals and $J(R) \neq 0$, then $J(R) \cap R^d \neq 0$ in all of the following cases:*

- (1) d is a derivation and R^d satisfies the acc on right annihilators of powers,
- (2) σ has locally finite order, R is an algebra over an uncountable field, and R^d satisfies the acc on right annihilators of powers,
- (3) σ has locally finite order and R^d satisfies the acc on right annihilators,
- (4) σ has locally finite order and R^d is reduced,
- (5) σ has locally finite order and R^d satisfies a polynomial identity.

Dowód. The beginning of this proof is that same as the beginning of the proof of Lemma 4. We let n denote the smallest degree of a nonzero element of $J(R)$ and if $n = 0$, there is nothing to prove. By way of contradiction, we will assume that $n > 0$ and let A be the elements of $J(R)$ of degree n . As in Lemma 4, the set $d^n(A) \cup \{0\}$ is a nonzero ideal of R^d .

If we let $S = d^n(A) \cup \{0\}$, then S is not nil. When we are in case (1), $\sigma = 1$, hence $S = S^\sigma$ and S^σ is not nil. When we are in cases (2), (3), (4), or (5), then Lemma 7 asserts that S^σ is not nil. Therefore, there exists some $a \in A$ such that $d^n(a)$ is not nilpotent and $\sigma(d^n(a)) = d^n(a)$.

Since $a \in J(R)$, $a + 1$ is invertible in R . Therefore, for every $i \geq 0$, then exists $b_i \in R$ such that

$$(7) \quad (a + 1)b_i = (d^n(a))^i$$

From among all the b_i , let m be such that b_m has minimal degree and let k be the degree of b_m .

From equation (7), we know that $(d^n(a))^m = (a + 1)b_m$. If we apply d^{n+k} to this equation and use the facts that a has degree $n > 0$, b_m has degree k , and $(d^n(a))^m$ has degree 0, we obtain

$$\begin{aligned} 0 &= d^{n+k}((d^n(a))^m) = d^{n+k}((a + 1)b_m) = d^{n+k}(ab_m) + d^{n+k}(b_m) = \\ &= d^{n+k}(ab_m) = \binom{n+k}{n} \sigma^k(d^n(a)) d^k(b_m) = \binom{n+k}{n} d^n(a) d^k(b_m). \end{aligned}$$

As a result, $d^n(a)d^k(b_m) = 0$.

Since $d^n(a)d^k(b_m) = 0$, if we apply d^k to $d^n(a)b_m$, we obtain

$$d^k(d^n(a)b_m) = \sigma^k(d^n(a))d^k(b_m) = d^n(a)d^k(b_m) = 0.$$

Therefore the degree of $d^n(a)b_m$ is less than k .

Next, since $(a + 1)b_m = (d^n(a))^m$, we can multiply this equation on the left by $d^n(a)$ to obtain

$$(8) \quad d^n(a)(a + 1)b_m = (d^n(a))^{m+1}.$$

On the other hand, if we apply d^n to $d^n(a)a - ad^n(a)$, we have

$$\begin{aligned} d^n(d^n(a)a - ad^n(a)) &= \sigma^n(d^n(a))d^n(a) - d^n(a)d^n(a) \\ &= d^n(a)d^n(a) - d^n(a)d^n(a) = 0. \end{aligned}$$

Thus $d^n(a)a - ad^n(a)$ is an element of $J(R)$ of degree less than n , hence $d^n(a)a - ad^n(a) = 0$. Since we now know that $d^n(a)$ and a commute, equation (8) can be rewritten as

$$(9) \quad (a + 1)(d^n(a)b_m) = (d^n(a))^{m+1}.$$

If we compare equations (7) and (9), we can see that $d^n(a)b_m$ and b_m both have the property that they produce a power of $d^n(a)$ when multiplied on the left by $a + 1$. However, the degree of $d^n(a)b_m$ is smaller than the degree of b_m , contradicting the minimality of the degree of b_m , thereby concluding the proof. \square

Our final lemma will play the role in proving Theorems 10 and 11 that Lemma 1 played in proving Theorems 5 and 6.

Lemma 9. *Let R be an algebra of characteristic 0 with a locally nilpotent σ -derivation d such that $d\sigma = \sigma d$. If $a \in (J(R) \cap R^d \cap d(R))^\sigma$ such that $\text{r. ann}_{R^d}(a) = \text{r. ann}_{R^d}(a^2)$, then $a = 0$.*

Dowód. Let $x \in R$ such that $d(x) = a$ and observe that

$$a = \sigma(a) = \sigma(d(x)) = d(\sigma(x)).$$

Since $xa \in J(R)$, xa has a quasi-inverse $b \in R$ and we have $xa + b = xab$. Multiplying this equation on the left by a we obtain

$$(10) \quad axa + ab = axab.$$

If $ab = 0$, then $axa = 0$. Applying d to both sides of this equation gives us $a^3 = 0$. However, since $\text{r. ann}_{R^d}(a) = \text{r. ann}_{R^d}(a^2)$, it follows that $a = 0$.

To conclude the proof, by way of contradiction, we may assume that $ab \neq 0$. If we let k denote the degree of ab and if $k \geq 1$, then applying d^{k+1} to both sides of equation (10) gives us

$$0 = d^{k+1}(axab) = (k + 1)\sigma^k(ad(x))d^k(ab) = (k + 1)a^3d^k(b).$$

Therefore $a^3d^k(b) = 0$, which implies that $ad^k(b) = 0$. As a result,

$$d^k(ab) = \sigma^k(a)d^k(b) = ad^k(b) = 0,$$

contradicting that the degree of ab is k .

The only remaining possibility is that $k = 0$. However, in this case, if we apply d to both sides of equation (10), we obtain $d(axa) = d(axab)$. Simplifying this equation gives us

$a^3 = a^3b$, which implies that $a = ab$. In light of this, equation (10) simplifies to $xa + b = xa$, which leads to the contradiction $b = 0$. Thus $a = 0$, as required. \square

We can now prove our final main result.

Theorem 10. *Let R be an algebra of characteristic 0 with a locally nilpotent regular σ -derivation d such that $d\sigma = \sigma d$. If R^d has no nonzero nil ideals, then $J(R) = 0$ in all of the following cases:*

- (1) d is a derivation and R^d satisfies the acc on right annihilators of powers,
- (2) σ has locally finite order, R is an algebra over an uncountable field, and R^d satisfies the acc on right annihilators of powers,
- (3) σ has locally finite order and R^d satisfies the acc on right annihilators,
- (4) σ has locally finite order and R^d is reduced,
- (5) σ has locally finite order and R^d satisfies a polynomial identity.

Dowód. By way of contradiction, suppose that $J(R) \neq 0$. Regardless of which case we are in, we can apply Lemma 8 to see that $J(R) \cap R^d \neq 0$. Since d is regular, $d(R) \cap R^d$ has nonzero intersection with $J(R) \cap R^d \neq 0$, hence

$$J(R) \cap R^d \cap d(R) = (J(R) \cap R^d) \cap (d(R) \cap R^d)$$

is a nonzero ideal of R^d .

In the first four cases, R^d satisfies the acc on right annihilators of powers. Therefore, if $\alpha \in (J(R) \cap R^d \cap d(R))^\sigma$, there exists $n \geq 1$ such that $\text{r. ann}_{R^d}(\alpha^n) = \text{r. ann}_{R^d}(\alpha^{n+1})$. If we let $a = \alpha^n$, then $\text{r. ann}_{R^d}(a) = \text{r. ann}_{R^d}(a^2)$ and Lemma 9 asserts that $a = 0$. As a result, $\alpha^n = a = 0$, hence every element of $(J(R) \cap R^d \cap d(R))^\sigma$ is nilpotent. Lemma 7 now asserts that $J(R) \cap R^d \cap d(R)$ contains a nonzero nil ideal. However, this immediately leads to the contradiction that R^d contains a nonzero nil ideal.

Finally, in case (5), since R^d is a semiprime ring satisfying a polynomial identity, it follows that every nonzero ideal of R^d has nonzero intersection with $Z(R^d)$, the center of R^d . Observe that $Z(R^d)$ is reduced, therefore if we let $S = (J(R) \cap R^d \cap d(R)) \cap Z(R^d)$, then $S \neq 0$ and Lemma 7 tells us that S^σ is not nil. Since R^d is semiprime, we know that if $a \in Z(R^d)$ then $\text{r. ann}_{R^d}(a) = \text{r. ann}_{R^d}(a^2)$. Therefore, if $0 \neq a \in S^\sigma$, then Lemma 9 provides us with the contradiction $a = 0$, thereby concluding the proof. \square

We conclude this paper by removing the condition the d is regular in Theorem 10 and examining the situation where $J(R^d) = 0$.

Theorem 11. *Let R be an algebra of characteristic 0 with a locally nilpotent σ -derivation d such that $d\sigma = \sigma d$. If $J(R^d) = 0$ then $J(R) = 0$ in all of the following cases:*

- (1) d is a derivation and R^d satisfies the acc on right annihilators of powers,
- (2) σ has locally finite order, R is an algebra over an uncountable field, and R^d satisfies the acc on right annihilators of powers,
- (3) σ has locally finite order and R^d satisfies the acc on right annihilators,
- (4) σ has locally finite order and R^d is reduced,
- (5) σ has locally finite order and R^d satisfies a polynomial identity.

Dowód. This result follows from Theorem 10 in the same fashion that Theorem 6 follows from Theorem 5. If we let A, B be the ideals of R constructed in the proof of Theorem 6, we can restrict the action of d to A and then apply Theorem 10 to conclude that $J(A) = 0$. Since $B \subseteq R^d$ and $A \oplus B$ is essential in R , we first see that $J(B) = 0$, then $J(A \oplus B) = J(A) \oplus J(B) = 0$, and finally, $J(R) = 0$. \square

LITERATURA

- [1] S.A. Amitsur, *Radicals of polynomial rings*, Canadian J. Math., 8 (1956), 355-361.
- [2] S.S. Bedi, J. Ram, *Jacobson radical of skew polynomial rings and skew group rings*, Israel J. Math., 35 (4) (1980), 327-338.
- [3] J. Bergen, P. Grzeszczuk, *On rings with locally nilpotent skew derivations*, Comm. in Algebra, 39 (2011), 3698-3708.
- [4] J. Bergen, P. Grzeszczuk, *Goldie dimension of constants of locally nilpotent skew derivations*, J. Algebra and its Applications, to appear.
- [5] J. Bergen, S. Montgomery, D.S. Passman, *Radicals of crossed products of enveloping algebras*, Israel J. Math., 59 (2) (1987), 167-184.
- [6] G.M. Bergman, I.M. Isaacs, *Rings with fixed-point-free group actions*, Proc. London Math. Soc. 27 (3) (1973), 69-87.
- [7] Y-T Tsai, T-Y Wu, C-L Chuang, *Jacobson radicals of Ore extensions of derivation type*, Comm. in Algebra, 35 (3) (2007), 975-982.

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