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GOLDIE DIMENSION OF CONSTANTS OF LOCALLY NILPOTENT SKEW DERIVATIONS

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Abstract. In this paper, we examine rings R with locally nilpotent skew derivations d and compare the Goldie dimension of R to that of the subring of constants R^d. This generalizes the situation where one compares the Goldie dimension of an Ore extension to that of the base ring. Under certain natural conditions placed upon R^d, we show that R and R^d have the same Goldie dimension.

1. Introduction

There has been a good deal of interest in the relationship between the Goldie dimension of a ring and that of naturally occurring ring extensions and subrings. In [7], it is proved that $\dim S = \dim S[x]$. More generally, the Goldie dimension of S and Ore extensions $S[x;\delta]$ were studied in [1], [3], [6], and extensions of skew derivation type were examined in [5].

If R is the q-skew Ore extension $R = S[x;\sigma,\delta]$, then the formula $\sigma(x) = q^{-1}x$ extends the automorphism σ to R and there is a q^{-1}-skew σ^{-1}-derivation $d\colon R \to R$ defined as $d(x) = 1$ and $d(a) = 0$ for $a \in S$ (cf. [2]). Observe that d is locally nilpotent and if q is either not a root of unity or $q = 1$ and S is of characteristic zero, then the subring of constants R^d is equal to S. Therefore, we can think of the relationship between a ring and an Ore extension as being a special case of the relationship between the subring of constants of a locally nilpotent q-skew derivation and the original algebra.

The concepts of a σ-derivation being regular and a ring being specially homogeneous will be defined immediately after the proofs of Proposition 1 and Lemma 7, respectively. Using these concepts, our two main results will be

Theorem 13. *Let d be a locally nilpotent q-skew σ-derivation of R, where q is not a root of unity or R has characteristic 0 and $q = 1$, such that*

1. *d is regular,*
2. *R is specially homogeneous.*

Then R has finite Goldie dimension if and only if R^d has finite Goldie dimension and $\dim_R R = \dim_{R^d} R^d$.

Corollary 14. *Let d be a locally nilpotent q-skew σ-derivation of an algebra R where q is not a root of unity or R has characteristic 0 and $q = 1$. If*

1. *R^d is σ-semiprime and*
2. *R^d is nonsingular,*

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then R has finite Goldie dimension if and only if R^d has finite Goldie dimension and $\dim_R R = \dim_{R^d} R^d$.

To put these results in perspective, Bell and Goodearl constructed in [1] a $\mathbb{Q}$-algebra S with a derivation δ, such that $\dim S = 1$ but $\dim S[x;\delta] = \infty$. Based upon our earlier observation, we can view this as an example of an algebra R with a locally nilpotent derivation d such that $\dim R^d = 1$ and $\dim R = \infty$. Furthermore, in this example, the derivation d is regular.

To illustrate the opposite point, let R be the Grassmann algebra over $\mathbb{Q}$ generated by $e_1, e_2, \ldots$. Next, let d be the $\mathbb{Q}$-linear derivation of R defined as $d(e_i) = e_{i-1}$, for $i > 1$, and $d(e_1) = 0$. In this case, d is locally nilpotent and R^d is spanned over $\mathbb{Q}$ by 1 and all products of the form $e_1 \cdots e_m$, where $m \geq 1$. Observe that

$$\mathbb{Q}e_1 \oplus \mathbb{Q}e_1e_2 \oplus \mathbb{Q}e_1e_2e_3 \oplus \cdots$$

is an infinite direct sum of left ideals of R^d, hence $\dim R^d = \infty$. On the other hand, if L_1, L_2 are nonzero left ideals of R, then there exists $n \in \mathbb{N}$ such that $e_1e_2\cdots e_n \in L_1 \cap L_2$. Thus $\dim R = 1$.

In light of these two examples, we can see that if either R^d or R has finite Goldie dimension then some additional assumptions, such as those in Theorem 13 and Corollary 14, are needed to show that R^d and R have the same Goldie dimension.

We should also point out that there is a strong relationship between the Goldie dimensions of R^d and R when R is semiprime and d is algebraic. In particular, it is proved in [4] that if R is semiprime and $d^n = 0$, then $\dim R^d \leqslant \dim R \leqslant n \cdot \dim R^d$.

We will now introduce the terminology and notation that will be used throughout the paper. R will be an algebra over a field K. If σ is a K-linear automorphism of R, then a σ-derivation d is a K-linear map $d : R \to R$ such that

$$d(rs) = d(r)s + \sigma(r)d(s),$$

for all $r, s \in R$. The ring of constants R^d is defined as

$$R^d = \{r \in R \mid d(r) = 0\}.$$

A σ-derivation d is said to be *locally nilpotent* if for every $r \in R$, there exists $n = n(r) \geq 1$ such that $d^n(r) = 0$. If q is a nonzero element of K, we say that our σ-derivation is *q-skew* if

$$d\sigma(r) = q\sigma d(r),$$

for all $r \in R$. For $n \geq 1$, let

$$(n!)_q = \prod_{k=1}^{n} (1 + q + \cdots + q^{k-1}).$$

Then the q-binomial coefficient $\binom{n}{i}_q$ is defined as evaluation at $t = q$ of the polynomial function

$$\binom{n}{i}_t = \frac{(t^n - 1)(t^{n-1} - 1)\ldots(t^{n-i+1} - 1)}{(t^i - 1)(t^{i-1} - 1)\ldots(t - 1)}.$$

If q is not a root of unity, then

$$\binom{n}{k}_q = \frac{(n!)_q}{((n-k)!)_q(k!)_q}$$

is nonzero for all $n \geqslant k \geqslant 0$.

The following q-Leibniz Rule holds in a ring with q-skew σ-derivation d.

$$d^n(ab) = \sum_{j=0}^{n} \binom{n}{j}_q \sigma^{n-j} d^j(a) d^{n-j}(b)$$

for all $a, b \in R$ and $n \geqslant 0$.

For $m \geqslant 0$, let $R_m = \ker d^{m+1}$. Clearly, d is locally nilpotent if and only if $R = \bigcup_{m \geqslant 0} R_m$. By the **degree** of an element $a \in R$, which we denote as $\deg(a)$, we mean the integer n such that $a \in R_n$ but $a \notin R_{n-1}$. The q-Leibniz Rule implies that $R_n R_m \subseteq R_{n+m}$, so R is a filtered algebra, with $R_0 = R^d$.

We will always assume that $1 + q + \cdots + q^m \neq 0$, for any integer $m \geqslant 1$. This means that either q is not a root of unity, or $q = 1$ and $\operatorname{char} K = 0$.

2. Results

Since d is locally nilpotent, any nonzero d-stable subset of R has nonzero intersection with R^d. Therefore, if E is a nonzero d-stable and σ-stable right (or left) ideal of R, then $E^d = E \cap R^d$ is a nonzero σ-stable right (or left) ideal of R. Throughout much of this section, R^d will be σ-semiprime. In this situation, it then follows that R has no nonzero one-sided ideals which are d-stable and σ-stable whose intersection with R^d is nilpotent.

Proposition 1. *Let d be a locally nilpotent q-skew σ-derivation of a ring R. If the subring of constants R^d is σ-semiprime, then there exist ideals A and B of R which are d-stable and σ-stable such that*

(1) $A \cap B = 0$ *and* $\operatorname{r.ann}_R(A \oplus B) = 0$,
(2) $B \subseteq R^d$,
(3) $d(A) \subseteq A$ *and* $\operatorname{r.ann}_A(d(A) \cap A^d) = 0$.

Proof. Let $C = \operatorname{r.ann}_R(d(R) \cap R^d)$. If $C = 0$, then it is enough to let $A = R$ and $B = 0$. Now suppose that $C \neq 0$; it is clear that C is a right ideal of R that is d-stable and σ-stable such that

$$(d(C) \cap C^d)^2 \subseteq (d(R) \cap R^d) \cdot C = 0.$$

The σ-semiprimeness of R^d implies that $d(C) \cap C^d = 0$. Furthermore, since d is locally nilpotent, we immediately see that $d(C) = 0$.

Next, let $B = RC$; since $Cd(R) = d(CR) \subseteq d(C) = 0$, we see that $Cd(R) \subseteq d(R) \cap R^d$. It now follows that

$$(d(R)C)^2 \subseteq d(R) \cdot (Cd(R)) \cdot C = 0.$$

Since $d(R)C$ is a σ-stable right ideal of R^d, we have $d(R)C = 0$ and so, $B = RC \subseteq R^d$.

Now let $A = \operatorname{r.ann}_R(B)$; clearly A and B are both σ-stable and d-stable ideals of R such that $(A \cap B)^2 \subseteq BA = 0$. It follows by the σ-semiprimeness of R^d that $A \cap B = 0$. In addition, since $\operatorname{r.ann}_R(A \oplus B) \subseteq \operatorname{r.ann}_R(B) = A$, we have

$$(\operatorname{r.ann}_R(A \oplus B))^2 \subseteq A \cdot \operatorname{r.ann}_R(A \oplus B) = 0.$$

The σ-semiprimeness of R^d now tells us that $\operatorname{r.ann}_R(A \oplus B) = 0$.

Finally, suppose that $0 \neq X = \mathrm{r.ann}_A(d(A) \cap A^d)$. Then X is a nonzero right ideal of R that is d-stable and σ-stable. Clearly $(d(A) \cap A^d)X^d = 0$ and so,

$$(d(R) \cap R^d) \cdot (X^d)^2 \subseteq (d(R) \cap R^d) \cdot A^d \cdot X^d \subseteq (d(A) \cap A^d) \cdot X^d = 0.$$

As a result,

$$(X^d)^2 \subseteq A \cap \mathrm{r.ann}_R(d(R) \cap R^d) = A \cap C \subseteq A \cap RC = A \cap B = 0.$$

Since $(X^d)^2 = 0$, the σ-semiprimeness of R^d implies that $X^d = 0$, hence $X = 0$. This is a contradiction, thereby proving (3). $\square$

Let d be a locally nilpotent σ-derivation of a ring R. In light of Proposition 1, it is natural to define d to be **right regular** (or simply **regular**) if $\mathrm{r.ann}_{R^d}(d(R) \cap R^d) = 0$. Observe that d being regular is equivalent to the condition that $\mathrm{r.ann}_R(d(R) \cap R^d) = 0$.

Observe that Proposition 1 asserts that if R^d is σ-semiprime, then R contains d-stable and σ-stable ideals A and B such that $A \cap B = 0$, $A \oplus B$ is essential in ${}_R R$, $B \subseteq R^d$, and d restricted to A is regular.

Lemma 2. *If $x_1, x_2, \ldots, x_n \in R_1$, then*

$$d^n(x_1 x_2 \ldots x_{n-1} x_n) = (n!)_q \sigma^{n-1} d(x_1) \sigma^{n-2} d(x_2) \ldots \sigma d(x_{n-1}) d(x_n).$$

Proof. Since each $x_i \in R_1$, it follows that $x_1 x_2 \ldots x_{n-1} \in R_{n-1}$. The q-Leibniz rule tells us that

$$d^n(x_1 x_2 \ldots x_{n-1} x_n) = \sum_{i=0}^{n} \binom{n}{i}_q \sigma^{n-i} d^i(x_1 x_2 \ldots x_{n-1}) d^{n-i}(x_n)$$
$$= \binom{n}{1}_q \sigma d^{n-1}(x_1 x_2 \ldots x_{n-1}) d(x_n).$$

The result follows now by induction. $\square$

For any $f \in R_n$ and $x_1, x_2, \ldots, x_n \in R_1$ we define the element

$$\widehat{f}(x_1, x_2, \ldots, x_n) = (n!)_q \sigma^{-1} d(x_1) \sigma^{-2} d(x_2) \ldots \sigma^{-n} d(x_n) f - x_1 x_2 \ldots x_n d^n(f).$$

Lemma 3. *For any $f \in R_n$ and $x_1, x_2, \ldots, x_n \in R_1$, the element $\widehat{f}(x_1, x_2, \ldots, x_n)$ has degree smaller than n.*

Proof. Since $(n!)_q \sigma^{-1} d(x_1) \sigma^{-2} d(x_2) \ldots \sigma^{-n} d(x_n) \in R^d$, applying Lemma 2, we have

$$d^n(\widehat{f}(x_1, x_2, \ldots, x_n)) = (n!)_q \sigma^n(\sigma^{-1} d(x_1) \sigma^{-2} d(x_2) \ldots \sigma^{-n} d(x_n)) d^n(f)$$
$$- d^n(x_1 x_2 \ldots x_n) d^n(f) = (n!)_q \sigma^{n-1} d(x_1) \sigma^{n-2} d(x_2) \ldots d(x_n) d^n(f)$$
$$- d^n(x_1 x_2 \ldots x_n) d^n(f) = 0.$$

$\square$

We continue with

Lemma 4. *Let d be a regular q-skew locally nilpotent σ-derivation of R. If $0 \neq f\alpha \in R^d \cap R\alpha$, where $\alpha \in R^d$ and $f \in R$, then there exists $\gamma \in R^d$ such that $0 \neq \gamma f\alpha \in R^d\alpha$.*

Proof. We will apply induction to the degree of f. If f has degree 0, then $0 \neq f \in R^d$. In this case, we can let $\gamma = 1$ and then $\gamma f\alpha = f\alpha \in R^d\alpha$.

Suppose that f has degree n and assume the result holds for elements of smaller degree. Since d is regular, $f\alpha$ does not annihilate $d(R) \cap R^d$ on the right. As a result, there exists $\gamma_n = d(x_n) \in d(R) \cap R^d$ such that $0 \neq \sigma^{-n}(\gamma_1)f\alpha$. Continuing as above, there exist $\gamma_{n-1} = d(x_{n-1}), \gamma_{n-2} = d(x_{n-2}), \dots, \gamma_1 = d(x_1) \in d(R) \cap R^d$ such that

$$(n!)_q\sigma^{-1}(\gamma_1)\sigma^{-2}(\gamma_2)\dots\sigma^{-(n-1)}(\gamma_{n-1})\sigma^{-n}(\gamma_n)f\alpha \neq 0.$$

Now consider the element $c = \widehat{f}(x_1, x_2, \dots, x_n)$; by Lemma 3, c has smaller degree than f. Recall that $0 = d(f\alpha) = d(f)\alpha$, therefore $d^n(f)\alpha = 0$ and

$$0 \neq c\alpha = (n!)_q\sigma^{-1}(\gamma_1)\sigma^{-2}(\gamma_2)\dots\sigma^{-n}(\gamma_n)f\alpha \in R\alpha \cap R^d.$$

The induction hypothesis now implies that there exists $\gamma_{n+1} \in R^d$ such that $0 \neq \gamma_{n+1}c\alpha \in R^d\alpha$.

However

$$\begin{aligned}\gamma_{n+1}c\alpha &= \gamma_{n+1}((n!)_q\sigma^{-1}(\gamma_1)\sigma^{-2}(\gamma_2)\dots\sigma^{-n}(\gamma_n)f - x_1x_2\dots x_nd^n(f))\alpha \\ &= (n!)_q\gamma_{n+1}\sigma^{-1}(\gamma_1)\sigma^{-2}(\gamma_2)\dots\sigma^{-n}(\gamma_n)f\alpha.\end{aligned}$$

Therefore, if we let $\gamma = (n!)_q\gamma_{n+1}\sigma^{-1}(\gamma_1)\sigma^{-2}(\gamma_2)\dots\sigma^{-n}(\gamma_n)$, it follows that $0 \neq \gamma f\alpha \in R^d\alpha$. □

The next result, along with the corollary that will follow it, proves one half of Theorem 13 and Corollary 14.

Theorem 5. *Let d be a regular q-skew locally nilpotent σ-derivation of an algebra R with finite Goldie dimension. If q is not a root of unity or R has characteristic 0 and $q = 1$, then the subalgebra of constants R^d has finite Goldie dimension and*

$$\dim_{R^d} R^d \leqslant \dim_R R.$$

Proof. It is enough to show that if $R^db_1 \oplus \dots \oplus R^db_n$ is a direct sum of left ideals of R^d then the sum $Rb_1 + \dots + Rb_n$ is also direct. We proceed by induction. Suppose that the sum $Rb_1 + \dots + Rb_k$ is direct and $(Rb_1 \oplus \dots \oplus Rb_k) \cap Rb_{k+1} \neq 0$, where $n > k \geqslant 1$.

Since the left ideals Rb_j are d-invariant, there exists $c \in R$ such that

$$0 \neq cb_{k+1} \in (Rb_1 \oplus \dots \oplus Rb_k)^d = (Rb_1)^d \oplus \dots \oplus (Rb_k)^d.$$

Thus $cb_{k+1} = r_1b_1 + \dots + r_kb_k$, where $r_1, \dots, r_k \in R$ and $r_1b_1, \dots, r_kb_k \in R^d$. Applying Lemma 4, there exist nonzero elements $\lambda_1, \dots, \lambda_k \in R^d$ such that if we let $\lambda = \lambda_k \dots \lambda_2\lambda_1$, we have

(1) $\lambda_1 r_1 b_1 \in R^d b_1, \lambda_2\lambda_1 r_2 b_2 \in R^d b_2, \dots, (\lambda_k \dots \lambda_2\lambda_1) r_k b_k \in R^d b_k$,
(2) not all elements from the chain $\lambda r_1 b_1, \lambda r_2 b_2, \dots \lambda r_k b_k$ are zero.

Thus $0 \neq \lambda c b_{k+1} = \sum_{j=1}^{k} \lambda r_j b_j \in R^d b_1 \oplus \cdots \oplus R^d b_k$.

Applying Lemma 4 once again, there exists $\gamma \in R^d$ such that $0 \neq \gamma\lambda c b_{k+1} \in R^d b_{k+1}$, so

$$0 \neq \gamma\lambda c b_{k+1} \in (R^d b_1 \oplus \cdots \oplus R^d b_k) \cap R^d b_{k+1},$$

which contradicts the assumption that the sum $R^d b_1 + \cdots + R^d b_n$ is direct. □

We can now use Proposition 1 to extend Theorem 5.

Corollary 6. *Let d be a q-skew locally nilpotent σ-derivation of R such that q is not a root of unity or R has characteristic 0 and $q = 1$. If the subalgebra of constants R^d is σ-semiprime and R has a finite Goldie dimension, then the Goldie dimension of R^d is finite and*

$$\dim_{R^d} R^d \leqslant \dim_R R.$$

Proof. Suppose S is a ring containing a direct sum of ideals $V \oplus W$ such that $\mathrm{r.ann}_S(V \oplus W) = 0$. In this situation, $\dim_S S = \dim_V V + \dim_W W$. If we let A, B be the ideals of R constructed in Proposition 1, it follows that $\dim_R R = \dim_A A + \dim_B B$.

Next, let $I = \mathrm{r.ann}_{R^d}(A^d \oplus B^d)$. Since $B = B^d$, it follows that $BI = 0$, hence $I \subseteq A$. As a result, $I \subseteq A^d \cap \mathrm{r.ann}_{R^d}(A^d)$. Therefore I is a σ-stable ideal of R^d of square 0 and the σ-semiprimeness of R^d implies that $I = 0$. Our observation in the previous paragraph now implies that $\dim_{R^d} R^d = \dim_{A^d} A^d + \dim_{B^d} B^d$. Since $\dim_B B = \dim_{B^d} B^d$, in order to prove our result, it suffices to show that $\dim_{A^d} A^d \leqslant \dim_A A$.

Proposition 1 showed that the restriction of d to A is regular. Since $\dim_A A \leqslant \dim_R R$, we know that $\dim_A A$ is finite, therefore we can apply Theorem 5 to conclude that $\dim_{A^d} A^d \leqslant \dim_A A$. □

We now begin the work needed to prove the second half of Theorem 13 and Corollary 14. For any $a \in R$, if we let $n = \deg(a)$, then it is clear that

$$\begin{aligned} \mathrm{l.ann}_{R^d}(a) &\subseteq \sigma^{-1}(\mathrm{l.ann}_{R^d}(d(a))) \subseteq \cdots \subseteq \sigma^{-n}(\mathrm{l.ann}_{R^d}(d^n(a))) \\ &\subseteq \sigma^{-n-1}(\mathrm{l.ann}_{R^d}(d^{n+1}(a))) = R^d. \end{aligned}$$

Lemma 7. *If $0 \neq a \in R$, then there exists $\lambda \in R^d$ such that $\lambda a \neq 0$ and*

$$\mathrm{l.ann}_{R^d}(\lambda a) = \sigma^{-1}(\mathrm{l.ann}_{R^d}(d(\lambda a))) = \cdots = \sigma^{-n}(\mathrm{l.ann}_{R^d}(d^n(\lambda a))),$$

where $n = \deg(\lambda a)$.

Proof. Let $n = \min\{\deg(\lambda a) \mid \lambda \in R^d \;\&\; \lambda a \neq 0\}$ and choose $\lambda \in R^d$ such that $\deg(\lambda a) = n$. By the observation before this lemma, it suffices to show that $\sigma^{-n}(\mathrm{l.ann}_{R^d}(d^n(\lambda a))) \subseteq \mathrm{l.ann}_{R^d}(\lambda a)$.

If $\gamma \in \sigma^{-n}(\mathrm{l.ann}_{R^d}(d^n(\lambda a)))$, we need to show that $\gamma \in \mathrm{l.ann}_{R^d}(\lambda a)$. Observe that

$$0 = \sigma^n(\gamma) d^n(\lambda a) = d^n(\gamma\lambda a),$$

so $\deg(\gamma\lambda a) \leqslant n - 1$. By the minimality of n, we see that $\gamma\lambda a = 0$, hence $\gamma \in \mathrm{l.ann}_{R^d}(\lambda a)$. □

We say that an element $a \in R$ of degree n is **special** if

$$\mathrm{l.ann}_{R^d}(a) = \sigma^{-1}(\mathrm{l.ann}_{R^d}(d(a))) = \cdots = \sigma^{-n}(\mathrm{l.ann}_{R^d}(d^n(a))),$$

where $n = \deg(a)$. Let $\mathcal{S}_n$ denote the set of all special elements of degree n. Observe that the proof of Lemma 7 showed that the nonzero elements of minimal degree in any left R^d-submodule of R are special.

A ring R with a q-skew locally nilpotent σ-derivation d is said to be **specially homogeneous** if, for any nonzero $a \in R$,

$$Ra \cap \mathcal{S}_n \neq \emptyset \implies Ra \cap \mathcal{S}_k \neq \emptyset \text{ for all } k \geqslant n.$$

In this case, any principal left ideal Ra must contain special elements of degree k, for all $k \geqslant \deg(a)$.

Proposition 8. *If the subalgebra of constants R^d is left nonsingular, then R is specially homogeneous.*

Proof. It is enough to show that if a nonzero element $a \in R$ is special of degree $n \geqslant 0$, then there exists an element $r \in R$ such that $ra \in \mathcal{S}_{n+1}$. Since d is regular, there exists a nonzero element $c = d(x) \in R^d$, such that $d(x)a \neq 0$. Using that a is special, we see that $\sigma^n d(x)d^n(a) \neq 0$ and it follows that

$$\begin{aligned} d^{n+1}(xa) &= \sum_{j=0}^{n+1} \binom{n+1}{j}_q \sigma^{n+1-j}(d^j(x))d^{n+1-j}(a) \\ &= \binom{n+1}{1}_q \sigma^n d(x)d^n(a) \neq 0. \end{aligned}$$

Hence $\deg(xa) = n + 1$.

Since R^d is left nonsingular and $0 \neq \sigma^n d(x)d^n(a) \in R^d$, we know that $\mathrm{l.ann}_{R^d}(\sigma^n d(x)d^n(a))$ is not essential in R^d. Therefore there exists a nonzero left ideal L of R^d such that $L \cap \mathrm{l.ann}_{R^d}(\sigma^n d(x)d^n(a)) = 0$. Notice that for any nonzero $l \in L$, $l\sigma^n d(x)d^n(a) \neq 0$. Therefore

$$d^{n+1}(\sigma^{-n-1}(l)xa) = ld^{n+1}(xa) = \binom{n+1}{1}_q l\sigma^n d(x)d^n(a) \neq 0,$$

hence $\deg(\sigma^{-n-1}(l)xa) = n + 1$.

By Lemma 7, there exists a nonzero $\lambda \in R^d$, such that $\lambda\sigma^{-n-1}(l)xa$ is special. It now suffices to show that $\lambda\sigma^{-n-1}(l)xa$ has degree $n + 1$. From the above, it now follows that

$$d^{n+1}(\lambda\sigma^{-n-1}(l)xa) = \binom{n+1}{1}_q \sigma^{n+1}(\lambda)l\sigma^n d(x)d^n(a).$$

As a result, $\deg(\lambda\sigma^{-n-1}(l)xa) < n + 1$ if and only if $\sigma^{n+1}(\lambda)l\sigma^n d(x)d^n(a) = 0$. However, since $\sigma^{n+1}(\lambda)l \in L$, the only way it can annihilate $\sigma^n d(x)d^n(a)$ on the left is for it to be 0. Thus $\sigma^{n+1}(\lambda)l = 0$, which implies that $\lambda\sigma^{-n-1}(l) = 0$. But this contradicts our assumption that $\lambda\sigma^{-n-1}(l)xa$ is special, hence $\lambda\sigma^{-n-1}(l)xa$ is indeed a special element of degree $n + 1$. $\square$

If we now let $S = d(R) \cap R^d$, recall that d being regular means that $\mathrm{r.ann}_{R^d}(S) = 0$.

Lemma 9. *If $a \in R$ is special and $t \in \mathrm{l.ann}_R(a)$, then there exists an integer $m = m(t) \geqslant 1$ such that $S^m t \subseteq R \cdot \mathrm{l.ann}_{R^d}(a)$.*

Proof. Suppose that $\deg(a) = n$ and $\deg(t) = l$. Then $\deg(ta) \leqslant n+l$, hence $0 = d^{l+n}(ta) = \binom{l+n}{n}_q \sigma^n(d^l(t))d^n(a)$. Since $a \in \mathcal{S}_n$ and $\sigma^n(d^l(t)) \in \mathrm{l.ann}_{R^d}(d^n(a))$, we have $d^l(t)a = 0$.

Given $s_1, s_2, \ldots, s_l \in S$, there exist $x_1, x_2, \ldots, x_l \in R_1$ such that $s_j = \sigma^{-j}(d(x_j))$, for $j = 1, 2, \ldots, l$. By Lemma 3, the element

$$\widehat{t} = (l!)_q \sigma^{-1} d(x_1)\sigma^{-2} d(x_2) \ldots \sigma^{-l} d(x_l)t - x_1 x_2 \ldots x_n d^l(t)$$

has degree smaller than l and it is clear that $\widehat{t}a = 0$. By induction on $\deg(t)$, there is an integer $\widehat{m}$ such that $S^{\widehat{m}}\widehat{t} \subseteq R \cdot \mathrm{l.ann}_{R^d}(a)$. We now have

$$(l!)_q s_1 s_2 \ldots s_l t = \widehat{t} + x_1 x_2 \ldots x_l d^l(t),$$

hence

$$S^{\widehat{m}+l} t \subseteq S^{\widehat{m}}\widehat{t} + R \cdot \mathrm{l.ann}_{R^d}(a) \subseteq R \cdot \mathrm{l.ann}_{R^d}(a).$$

□

We continue with

Corollary 10. *If a and b are special elements of R such that $\mathrm{l.ann}_{R^d}(a) = \mathrm{l.ann}_{R^d}(b)$, then $\mathrm{l.ann}_R(a) = \mathrm{l.ann}_R(b)$.*

Proof. It suffices to show that $\mathrm{l.ann}_R(a) \subseteq \mathrm{l.ann}_R(b)$ and, to this end, let $t \in \mathrm{l.ann}_R(a)$. By Lemma 9, there exists $m \geqslant 1$ such that

$$S^m t \subseteq R \cdot \mathrm{l.ann}_{R^d}(a) = R \cdot \mathrm{l.ann}_{R^d}(b).$$

As a result, $S^m tb = 0$. Since $\mathrm{r.ann}_{R^d}(S) = 0$, we also know that $\mathrm{r.ann}_R(S) = 0$, hence $tb = 0$. □

In our final two lemmas, we will assume that d is regular and R is specially homogeneous.

Lemma 11. *If L is an essential left ideal of R^d, then RL is an essential left ideal of R.*

Proof. Suppose not; if RL is not essential in R, let $a \in R$ be of minimal degree such that $RL \cap Ra = 0$. Observe that $a \in \mathcal{S}_n$, for some $n \geqslant 1$, and since $0 \neq d^n(a) \in R^d$, there exists $b \in R^d$ such that $0 \neq bd^n(a) \in L \cap Rd^n(a)$. By replacing a by $\sigma^{-n}(b)a$, without loss of generality, we may assume that $d^n(a) \in L$.

Let $r \in R$ such that $rd^n(a) \in \mathcal{S}_n$; then $0 \neq d^n(rd^n(a)) = d^n(r)d^n(a) \in R^d$. Therefore, by Lemma 4, there exists a nonzero element $\lambda \in R^d$ such that $0 \neq \lambda d^n(r)d^n(a) = \lambda^* d^n(a)$ for some $\lambda^* \in R^d$. Since

$$d^n(\sigma^{-n}(\lambda^*)a) = \lambda^* d^n(a) = d^n(\sigma^{-n}(\lambda)rd^n(a)),$$

it is clear that $\sigma^{-n}(\lambda^*)a$ and $\sigma^{-n}(\lambda)rd^n(a)$ are special of degree n and produce the same result when plugged into d^n. Therefore, they have the same left annihilator in R^d. Furthermore,

$$\deg(\sigma^{-n}(\lambda^*)a - \sigma^{-n}(\lambda)rd^n(a)) < n,$$

and, by the minimality of n, we have

$$RL \cap R \cdot (\sigma^{-n}(\lambda^*)a - \sigma^{-n}(\lambda)rd^n(a)) \neq 0.$$

Finally, let $s \in R$ be such that $0 \neq s(\sigma^{-n}(\lambda^*)a - \sigma^{-n}(\lambda)rd^n(a)) \in RL$. Since $d^n(a) \in L$, it follows that $s\sigma^{-n}(\lambda^*)a \in RL \cap Ra = 0$. By Corollary 10, $\sigma^{-n}(\lambda^*)a$ and $\sigma^{-n}(\lambda)rd^n(a)$ have the same left annihilator in R. Thus $s\sigma^{-n}(\lambda)rd^n(a) = 0$, which results the contradiction $s(\sigma^{-n}(\lambda^*)a - \sigma^{-n}(\lambda)rd^n(a)) = 0$. □

One additional lemma is required before we can prove our main results.

Lemma 12. *If $I = R^d a$ is a uniform left ideal of R^d, then $RI = Ra$ is a uniform left ideal of R.*

Proof. Suppose RI is not uniform; then from among all $x, y \in RI$ with $Rx \cap Ry = 0$, choose x, y such than $\deg(x) + \deg(y)$ minimal. Therefore x and y are special and, without loss of generality, we may assume that $\deg(x) \leqslant \deg(y)$. By Lemma 3, Rx contains a special element z such that $\deg(z) = \deg(y) = m$.

Since $d^m(z)$ and $d^m(y)$ are nonzero elements of $(RI)^d = (Ra)^d$, Lemma 4 asserts that there exist elements $\alpha, \beta \in R^d$ such that $0 \neq \alpha d^m(z) \in R^d a$ and $0 \neq \beta d^m(y) \in R^d a$. It follows, since $R^d a$ is uniform, that there exist $\lambda_1, \lambda_2 \in R^d$ such that $\lambda_1 \alpha d^m(z) = \lambda_2 \beta d^m(y) \neq 0$.

By replacing z by $\sigma^{-n}(\lambda_1\alpha)z$ and y by $\sigma^{-1}(\lambda_2\beta)y$, without loss, we may assume that $Rz \cap Ry = 0$, $\deg(z) = \deg(y)$, and $d^m(z) = d^m(y)$. Since z and y are special and produce the same result when plugged into d^n, they have the same left annihilator in R^d. Certainly $\deg(y - z) < \deg(y)$ and it now follows, by the minimality of $\deg(x) + \deg(y)$, that $Rx \cap R(y - z) \neq 0$. Next, let $r_1, r_2 \in R$ such that $r_1 x = r_2(y - z) \neq 0$. Thus $r_2 y = r_1 x + r_2 z \in Rx \cap Ry = 0$, hence $r_2 y = 0$. By Corollary 10, z and y have the same left annihilator in R, therefore $r_2 z = 0$. This immediately leads to the contradiction $r_1 x = 0$. □

We can now prove the first of our two main results.

Theorem 13. *Let d be a locally nilpotent q-skew σ-derivation of R, where q is not a root of unity or R has characteristic 0 and $q = 1$, such that*

(1) *d is regular,*
(2) *R is specially homogeneous.*

Then R has finite Goldie dimension if and only if R^d has finite Goldie dimension and $\dim_R R = \dim_{R^d} R^d$.

Proof. Observe that Theorem 5 covers one half of this result, while not requiring that R be specially homogeneous. For the other half, we will assume that R^d has finite Goldie dimension and we need to show that $\dim_R R = \dim_{R^d} R^d$. If we let $n = \dim_{R^d} R^d$, then there exist $a_i \in R^d$ such that $R^d a_1 \oplus \cdots \oplus R^d a_n$ is a direct sum of uniform left ideals of R^d which is also an essential left ideal of R^d.

It follows, from Lemmas 11 and 12, that $Ra_1 + \cdots + Ra_n$ is a sum of uniform left ideals of R which is an essential left ideal of R. Therefore, it suffices to show that the sum is actually direct. If the sum is not direct, then we can reorder the a_i such that there exist $r_i \in R$ with

$$0 \neq r_1a_1 = r_2a_2 + \cdots + r_na_n.$$

Applying Lemma 4, as in the proof of Theorem 5, there exists $\gamma \in R^d$ such that each $\gamma r_i \in R^d$ and

$$0 \neq \gamma r_1a_1 = \gamma r_2a_2 + \cdots + \gamma r_na_n.$$

However, this contradicts that the sum $R^da_1 + \cdots + R^da_n$ is direct, therefore $\dim_R R$ is also equal to n. □

We can now use Propositions 1 and 8 along with Theorem 13 to prove our second main result.

Corollary 14. *Let d be a locally nilpotent q-skew σ-derivation of an algebra R where q is not a root of unity or R has characteristic* 0 *and $q = 1$. If*

(1) *R^d is σ-semiprime and*
(2) *R^d is nonsingular,*

then R has finite Goldie dimension if and only if R^d has finite Goldie dimension and $\dim_R R = \dim_{R^d} R^d$.

Proof. Observe that Corollary 6 covers one half of this result, while not requiring that R^d be nonsingular. For the other half, we will assume that R^d has finite Goldie dimension and we need to show that $\dim_R R = \dim_{R^d} R^d$. As in the proof of Corollary 6, we can let A, B be the ideals constructed in Proposition 1 and it suffices to show that $\dim_A A = \dim_{A^d} A^d$.

When d is restricted to A, A^d is nonsingular and Proposition 8 tells us that A is specially homogeneous. Since $\dim_{R^d} R^d$ is finite, so is $\dim_{A^d} A^d$, therefore we can apply Theorem 13 to conclude that $\dim_A A = \dim_{A^d} A^d$. □

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