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SKEW DERIVATIONS AND THE NIL AND PRIME RADICALS

JEFFREY BERGEN AND PIOTR GRZESZCZUK

ABSTRACT. We examine when the nil and prime radicals of an algebra are stable under q-skew σ-derivations. We provide an example which shows that even if q is not a root of 1 or if δ and σ commute in characteristic 0, then the nil and prime radicals need not be δ-stable. However, when certain finiteness conditions are placed on δ or σ, then the nil and prime radicals are δ-stable.

In this paper, we examine when the nil and prime radicals of an algebra are stable under q-skew derivations. Throughout this paper, R will be an algebra over a field F. The nil radical of R will be denoted as $N(R)$ and it is the largest nil two-sided ideal of R. The prime radical of R will be denoted as $P(R)$ and is the intersection of all the prime ideals of R. It is well known that $P(R) \subseteq N(R)$ and that $\sigma(P(R)) = P(R)$ and $\sigma(N(R)) = N(R)$, for any automorphism σ of R. When F has characteristic 0, Proposition 2.6.28 of [R] shows that if δ is a derivation of R, then $\delta(N(R)) \subseteq N(R)$ and $\delta(P(R)) \subseteq P(R)$. Whenever f is a function and A is a subset of R such that $f(A) \subseteq A$, we say that A is f-stable. In [LMS], the authors examine various conditions under which the Jacobson radical is stable under actions of finite dimensional semisimple Hopf algebras.

For any prime p, the case when F has characteristic p is quite different. For example, let $R = F[x \mid x^p = 0]$ and consider the F-linear derivation δ defined as $\delta(x) = 1$. In this example, neither $N(R)$ nor $P(R)$ are δ-stable as $x \in P(R) \subseteq N(R)$ but $\delta(x) = 1 \notin N(R)$.

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If σ is an F-linear automorphism of R we say that δ is a σ-derivation if

$$\delta(rs) = \delta(r)s + \sigma(r)\delta(s),$$

for all $r, s \in R$. Furthermore, if $0 \neq q \in F$, we say that δ is a q-skew derivation provided

$$\delta(\sigma(r)) = q\sigma(\delta(r)),$$

for all $r \in R$. Regardless of the characteristic of F, the behavior of q-skew derivations when q is a root of 1 is often quite similar to that of derivations in characteristic p. For example, suppose $q^n = 1$ and $q \neq 1$. If we let $R = F[x \mid x^n = 0]$, then there is an automorphism σ such that $\sigma(x) = qx$ and a q-skew derivation δ such that $\delta(x) = 1$. Observe that neither $P(R)$ nor $N(R)$ are δ-stable as $x \in P(R) \subseteq N(R)$ but $\delta(x) = 1 \notin N(R)$. Note that since $1 + q + \cdots + q^{n-1} = 0$, δ preserves the relation $x^n = 0$.

In light of the above, it remains to consider the case where $1 + q + \cdots + q^{n-1} \neq 0$, for all $n \in \mathbb{N}$. This is equivalent to saying that either q is not a root of 1 or that $q = 1$ and F has characteristic 0. In this situation, the behavior of q-skew derivations is often quite similar to that of derivations in characteristic 0. However, we now present an example that shows that the nil and prime radicals need to be δ-stable in this situation. Following this example, we will show that when certain finiteness conditions are placed on σ or δ, the nil and prime radicals will be δ-stable.

The example below is motivated by an example in [BR] in which the authors examine the Jacobson radical of skew polynomial rings of automorphism type.

Example 1. *Let $0 \neq q \in F$ such that $1 + q + \cdots + q^{n-1} \neq 0$, for all $n \in \mathbb{N}$. Then there exists an F-algebra R with an automorphism σ and a locally nilpotent q-skew derivation δ such that neither the nil radical nor the prime radical of R are δ-stable.*

Proof. Let F be a field and let B be the set of all bi-infinite sequences of elements of F. Thus $B = \{(\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots) \mid a_i \in F\}$ and B is a ring where addition and multiplication are defined componentwise. Observe that B is commutative with no nonzero nilpotent elements. Next, let τ denote the right-shift operator on B, thus $\tau((\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots)) = (\ldots, b_{-2}, b_{-1}, b_0, b_1, b_2, \ldots)$, where $b_i = a_{i-1}$, for all $i \in \mathbb{Z}$. Note that τ is an automorphism of B

If we let A consist of the elements of B with only a finite number of nonzero entries, then A is an ideal of B. Now let $e = (\ldots, 1, 1, 1, 1, 1, \ldots)$

denote the multiplicative identity of B and let Fe be all multiples of e by elements of F. If we let $C = Fe + A$, then C is a commutative algebra over F with no nonzero nilpotent elements, τ is an automorphism of C, and A is a τ-stable ideal of C of codimension 1.

Let $R = C[x; \tau]$ be the skew polynomial ring over C of automorphism type. Therefore every element of R can be written uniquely as a finite sum of the form $\sum_{i=0}^{n} c_i x^i$. When multiplying in R, we have $xc = \tau(c)x$, for all $c \in C$. Let e_1 be the element of A where every component is 0 except the $i = 1$ component is 1. Thus e_1 has the properties that $0 \neq e_1 = (e_1)^2$ and $e_1 \tau^t(e_1) = 0$, for $t \neq 0$. If $c \in C$ and $n \geq 0$, then, computing in R, we have

$$\begin{aligned}(e_1x)(cx^n)(e_1x) &= (e_1x)(c\tau^n(e_1))x^{n+1} = e_1\tau(c\tau^n(e_1))x^{n+2} \\ &= (e_1\tau^{n+1}(e_1))\tau(c)x^{n+2} = 0.\end{aligned}$$

The previous equation tells us that $(R(e_1x)R)^2 = 0$, hence

$$e_1x \in R(e_1x)R \subseteq P(R) \subseteq N(R).$$

Next, we can define an automorphism σ of R as $\sigma(c) = \tau^{-1}(c)$, for all $c \in C$ and $\sigma(x) = qx$. Since $1 + q + \cdots + q^{n-1} \neq 0$, for all $n \in \mathbb{N}$, we can apply Theorem 2 of [BG] to conclude that there is a q-skew derivation δ of R such that $\delta(c) = 0$, for all $c \in C$, and $\delta(x) = 1$. Furthermore, Theorem 2 of [BG] also asserts that δ is locally nilpotent and its ring of constants, $R^\delta = \{r \in R \mid \delta(r) = 0\}$, is equal to C.

We know that $e_1x \in P(R) \subseteq N(R)$. However,

$$0 \neq \delta(e_1x) = \delta(e_1)x + \sigma(e_1)\delta(x) = \tau^{-1}(e_1) \in C.$$

Since C has no nonzero nilpotent elements, we see that $\delta(e_1x)$ is not nilpotent and cannot belong to $N(R)$. As a result, $e_1x \in P(R) \subseteq N(R)$ and $\delta(e_1x) \notin N(R)$ and so, the nil and prime radicals of R are not δ-stable. □

We now begin the work needed to show that if certain finiteness conditions are placed on σ or δ, then the assumption that $1 + q + \cdots + q^{n-1} \neq 0$, for all $n \in \mathbb{N}$ is enough to guarantee that $P(R)$ and $N(R)$ are δ-stable. Our earlier example indicates that it is not enough to assume that δ is locally nilpotent. If σ has locally finite order then $N(R)$ will be δ-stable and $P(R)$ will be δ-stable under the somewhat weaker condition that σ is locally algebraic. Both $P(R)$ and $N(R)$ will be δ-stable if we assume that δ is algebraic. In the next lemma, we will see that some of these assumptions place certain restriction on the possible values of q.

Lemma 2. *Let δ be a q-skew derivation of R where $1+q+\cdots+q^{n-1}\neq 0$, for all $n\in\mathbb{N}$.*

(i) If σ has locally finite order and $\delta\neq 0$, then $q=1$ and F has characteristic 0.

(ii) If δ is algebraic then either δ is nilpotent or $q=1$ and F has characteristic 0.

Proof. For (i), let $r\in R$ such that $\delta(r)\neq 0$. Since σ has locally finite order, there exists $n\in\mathbb{N}$ such that $\sigma^n(r)=r$ and $\sigma^n(\delta(r))=\delta(r)$. Observe that $\delta\sigma^n=q^n\sigma^n\delta$, therefore

$$\delta(r)=\sigma^n(\delta(r))=q^{-n}\delta(\sigma^n(r))=q^{-n}\delta(r).$$

Since $\delta(r)\neq 0$, we see that $q^n=1$. Furthermore, since $1+q+\cdots+q^{n-1}\neq 0$, we know that $q=1$, which immediately implies that F has characteristic 0.

For (ii), since δ is algebraic over F, there exists some minimal $n\in\mathbb{N}$ and $\alpha_i\in F$ such that

$$\delta^n(r)=\alpha_{n-1}\delta^{n-1}(r)+\cdots+\alpha_1\delta(r)+\alpha_0 r, \tag{1}$$

for all $r\in R$. If we replace r by $\sigma(r)$ in equation (1) and use the fact that $\delta^j\sigma=q^j\sigma\delta^j$, for all $j\in\mathbb{N}$, we obtain

$$q^n\sigma(\delta^n(r))=q^{n-1}\alpha_{n-1}\sigma(\delta^{n-1}(r))+\cdots+q\alpha_1\sigma(\delta(r))+\alpha_0\sigma(r).$$

Applying σ^{-1} to the previous equation and then multiplying by q^{-n} results in

$$\delta^n(r)=q^{-1}\alpha_{n-1}\delta^{n-1}(r)+\cdots+q^{1-n}\alpha_1\delta(r)+q^{-n}\alpha_0 r.$$

When we compare the previous equation to equation (1), the minimality of n asserts that $q^{n-i}\alpha_i=\alpha_i$, for $0\leq i\leq n-1$. If each $\alpha_i=0$, then δ is nilpotent. On the other hand, if some $\alpha_i\neq 0$, then q must be a root of 1. As in the proof of part (1), since q is a root of 1, it follows that $q=1$ and F has characteristic 0. □

Our next lemma does not require that δ be q-skew nor that R be an algebra.

Lemma 3. *Let R be a ring with a σ-derivation δ.*

(i) If I is a σ-stable ideal of R, then $I+\delta(I)$ is an ideal of R.

(ii) If $\delta(s)$ is nilpotent, for all $s\in N(R)$, then $N(R)$ is δ-stable.

Proof. For (i), if $r \in R$ and $s \in I$, then

$$r\delta(s) = \delta(\sigma^{-1}(r)s) - \delta(\sigma^{-1}(r))s \in I + \delta(I)$$

and

$$\delta(s)r = \delta(sr) - \sigma(s)\delta(r) \in I + \delta(I).$$

Therefore $R\delta(I), \delta(I)R \subseteq I + \delta(I)$ and so, $I + \delta(I)$ is an ideal of R. In particular, this tells us that $N(R) + \delta(N(R))$ is an ideal of R.

For (ii), if $r, s \in N(R)$ and $n \in \mathbb{N}$, then $(r + \delta(s))^n = (\delta(s))^n + w$, where $w \in N(R)$. Since $\delta(s)$ is nilpotent, we can choose n such that $(\delta(s))^n = 0$, hence $(r + \delta(s))^n \in N(R)$. As a result, $(r + \delta(s))^n$ is nilpotent which immediately implies that $r + \delta(s)$ is nilpotent. Therefore $N(R) + \delta(N(R))$ is a nil ideal, hence it must be contained in $N(R)$. Thus $\delta(N(R)) \subseteq N(R)$, as required. □

For the remainder of this paper, if $n \in \mathbb{N}$, we let

$$(n!)_q = (1)(1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{n-1}).$$

Note that if $q = 1$, then $(n!)_q = n!$.

Lemma 4. *Let R be a ring with q-skew derivation δ. If I is a σ-stable ideal of R and $r_1, r_2, \ldots, r_n \in I$, then*

(i) $\delta^n(r_1 r_2 \cdots r_n) = (n!)_q \sigma^{n-1}(\delta(r_1))\sigma^{n-2}(\delta(r_2)) \cdots \sigma(\delta(r_{n-1}))\delta(r_n) + w$, *where $w \in I$;*

(ii) $\sigma^{n-1}(\delta(r_1))\sigma^{n-2}(\delta(r_2)) \cdots \sigma(\delta(r_{n-1}))\delta(r_n) = q^{-\frac{(n-1)n}{2}}\delta(\sigma^{n-1}(r_1))\delta(\sigma^{n-2}(r_2)) \cdots \delta(\sigma(r_{n-1}))\delta(r_n)$;

(iii) if $\sigma(I) = I$, $(n!)_q \neq 0$, and $\delta^n(I^n) \subseteq K$, for some ideal K, then $(\delta(I))^n \subseteq I + K$.

Proof. For (i), if $r_1, \ldots, r_n \in R$, we have

$$\delta(r_1 r_2 \cdots r_{n-1} r_n) = \delta(r_1) r_2 \cdots r_{n-1} r_n + \sigma(r_1)\delta(r_2) \cdots r_{n-1} r_n + \cdots +$$

$$\sigma(r_1) \cdots \sigma(r_{n-2})\delta(r_{n-1}) r_n + \sigma(r_1)\sigma(r_2) \cdots \sigma(r_{n-1})\delta(r_n). \tag{2}$$

If $1 \leq k \leq n$, let $f_k = \sum_{i=0}^{n-k} \sigma^{n-k-i}\delta\sigma^i$. Repeated application of δ to equation (2) results in

$$\delta^n(r_1 r_2 \cdots r_n) = f_1(r_1) f_2(r_2) \cdots f_n(r_n) + w, \tag{3}$$

where w is a sum of terms of the form $g_1(r_1) g_2(r_2) \cdots \sigma^j(r_i) \cdots g_n(r_n)$ such that $j \geq 0$ and each g_i is a composition of l copies of δ and σ, for some $0 \leq l \leq n$.

Since δ is q-skew, it follows that $f_k(r) = (1 + q + \cdots + q^{n-k})\sigma^{n-k}(\delta(r))$, for all $r \in R$. Thus

$$f_1(r_1)f_2(r_2)\cdots f_n(r_n) =$$
$$(1)(1+q)\cdots(1+q+\cdots+q^{n-1})\sigma^{n-1}(\delta(r_1))\cdots\sigma(\delta(r_{n-1}))\delta(r_n) =$$
$$(n!)_q\sigma^{n-1}(\delta(r_1))\cdots\sigma(\delta(r_{n-1}))\delta(r_n).$$

Therefore, if each $r_i \in I$, we can rewrite equation (3) as

$$\delta^n(r_1r_2\cdots r_n) = (n!)_q\sigma^{n-1}(\delta(r_1))\sigma^{n-2}(\delta(r_2))\cdots\sigma(\delta(r_{n-1}))\delta(r_n) + w,$$

where $w \in I$, proving part (i).

Since $\delta\sigma = q\sigma\delta$, we see that $\sigma^{n-i}\delta = q^{-(n-i)}\delta\sigma^{n-i}$, for $1 \leq i \leq n$. Therefore part (ii) follows by replacing each term of the form $\sigma^{n-i}(\delta(r_i))$ in

$$\sigma^{n-1}(\delta(r_1))\sigma^{n-2}(\delta(r_2))\cdots\sigma(\delta(r_{n-1}))\delta(r_n)$$

by $q^{-(n-i)}\delta(\sigma^{n-i}(r_i))$.

For (iii), we know that both $(n!)_q$ and $q^{-\frac{(n-1)n}{2}}$ are nonzero. Therefore, since $\delta^n(I^n) \subseteq K$, it follows from parts (i) and (ii) that

$$\delta(\sigma^{n-1}(r_1))\delta(\sigma^{n-2}(r_2))\cdots\delta(\sigma(r_{n-1}))\delta(r_n) \in I + K. \tag{4}$$

In addition, $\sigma(I) = I$, thus $\sigma^i(I) = I$, for all $i \in \mathbb{N}$. It now follows from (4) that $(\delta(I))^n \subseteq I + K$. □

We can now prove

Theorem 5. *Let R be an algebra over a field of characteristic 0 with a σ-derivation δ such that δ and σ commute. If σ has locally finite order then the nil radical of R is δ-stable.*

Proof. Let $r \in N(R)$; in light of Lemma 3, it suffices to show that $\delta(r)$ is nilpotent. Since σ has locally finite order, there exists $n \in \mathbb{N}$ such that $\sigma^n(r) = r$ and we can let $s = \sigma^{-n+1}(r)\cdots\sigma^{-2}(r)\sigma^{-1}(r)r$. Note that $\sigma^{-n}(s) = s$ and, for any $m \in \mathbb{N}$, it now follows that

$$s^m = \sigma^{(1-m)n}(s)\cdots\sigma^{-2n}(s)\sigma^{-n}(s)s = \sigma^{1-mn}(r)\cdots\sigma^{-2}(r)\sigma^{-1}(r)r.$$

Since $s \in N(R)$, we can choose m such than $s^m = 0$ and we now have

$$0 = \delta^{mn}(s^m) = \delta^{mn}(\sigma^{1-mn}(r)\cdots\sigma^{-2}(r)\sigma^{-1}(r)r).$$

Observe that δ is q-skew with $q = 1$. Therefore $(n!)_q = n!$ and $\sigma^i\delta = \delta\sigma^i$, for all $i \in \mathbb{N}$. As a result, the term

$$(n!)_q\sigma^{n-1}(\delta(r_1))\sigma^{n-2}(\delta(r_2))\cdots\sigma(\delta(r_{n-1}))\delta(r_n)$$

in Lemma 4 can now be written as

$$n!\delta(\sigma^{n-1}(r_1))\delta(\sigma^{n-2}(r_2))\cdots\delta(\sigma(r_{n-1}))\delta(r_n).$$

Applying Lemma 4(i) with $I = N(R)$ now gives us

$$0 = \delta^{mn}(\sigma^{1-mn}(r)\cdots\sigma^{-2}(r)\sigma^{-1}(r)r) = (mn)!\delta(r)\cdots\delta(r)\delta(r)\delta(r) + w,$$

where $w \in N(R)$. Thus $(mn)!(\delta(r))^{mn} \in N(R)$ and, since F has characteristic 0, this immediately implies that $\delta(r)$ is nilpotent. □

For any ring S, let $W(S)$ be the sum of the nilpotent ideals of S. A useful property of the prime radical of R is that it can also be defined as the union of an ascending chain of ideals $P_\alpha \subseteq R$ as follows:

- $P_0 = 0$, $P_1 = W(R)$;
- $P_{\alpha+1}$ is the ideal of R such that $W(R/P_\alpha) = P_{\alpha+1}/P_\alpha$;
- if α is a limit ordinal, then $P_\alpha = \bigcup_{\beta<\alpha} P_\beta$.

Observe that each P_α is σ-stable and there exists an ordinal γ such that $P_\gamma = P_{\gamma+1} = P(R)$. For our next result, we can weaken the assumption in Theorem 5 and assume instead that σ is locally algebraic. This means that every element of R is contained in a finite dimensional σ-stable subspace of R.

Theorem 6. *Let R be an algebra with a q-skew derivation δ such that $1+q+\cdots+q^{n-1} \neq 0$, for all $n \in \mathbb{N}$. If σ is locally algebraic, then the prime radical of R is δ-stable.*

Proof. We will prove, using transfinite induction, that $\delta(P_\alpha) \subseteq P_\alpha$, for every ordinal α. To this end, suppose $\delta(P_\beta) \subseteq P_\beta$, for all ordinals $\beta < \alpha$. If α is a limit ordinal, we have

$$\delta(P_\alpha) = \delta(\bigcup_{\beta<\alpha} P_\beta) = \bigcup_{\beta<\alpha} \delta(P_\beta) \subseteq \bigcup_{\beta<\alpha} P_\beta = P_\alpha.$$

Next, suppose $\alpha = \beta + 1$ and let $a \in P_\alpha$; we will show that $\delta(a) \in P_\alpha$. Using that P_α is σ-stable, it follows that $(R\sigma^j(a)R + P_\beta)/P_\beta$ is a nilpotent ideal of R/P_β, for all $j \geq 0$. Since σ is locally algebraic, there exists $m \in \mathbb{N}$ such that

$$\sum_{j=0}^{\infty}(R\sigma^j(a)R + P_\beta)/P_\beta = \sum_{j=0}^{m}(R\sigma^j(a)R + P_\beta)/P_\beta.$$

Therefore, there exists an ideal J such that $\sigma(J) = J$, $a \in J$ and having the additional properties that

$$(RaR + P_\beta)/P_\beta \subseteq (J + P_\beta)/P_\beta$$

and $J^n \subseteq P_\beta$, for some $n > 0$. Since $\delta(P_\beta) \subseteq P_\beta$, we have $\delta^n(J^n) \subseteq \delta^n(P_\beta) \subseteq P_\beta$.

Applying Lemma 4(iii) with $I = J$, and $K = P_\beta$, we have $\delta(J)^n \subseteq J+P_\beta$. Thus $(J+\delta(J))^n \subseteq J+\delta(J)^n \subseteq J+P_\beta$, hence $(J+\delta(J))^{n^2} \subseteq (J+P_\beta)^n \subseteq P_\beta$. By Lemma 3(i), $J+\delta(J)$ is an ideal, therefore $J+\delta(J) \subseteq P_{\beta+1} = P_\alpha$. Since $a \in J$, we have $\delta(a) \in P_\alpha$. □

An ideal I is called a semiprime ideal if whenever J is an ideal and $n \in \mathbb{N}$ such that $J^n \subseteq I$, we have $J \subseteq I$. Observe that both $N(R)$ and $P(R)$ are semiprime ideals of R.

Theorem 7. *Let R be an algebra with a q-skew derivation δ such that δ is algebraic and $1+q+\cdots+q^{n-1} \neq 0$, for all $n \in \mathbb{N}$.*

(i) If I is a semiprime ideal of R such that $\sigma(I) = I$, then I is δ-stable.
(ii) The nil radical and prime radical of R are both δ-stable.

Proof. Since $N(R)$ and $P(R)$ are both semiprime ideals of R with $\sigma(N(R)) = N(R)$ and $\sigma(P(R)) = P(R)$, we see that part (ii) follows directly from part (i). Lemma 2(ii) showed that whenever δ is algebraic, either δ is nilpotent or $q = 1$ and F has characteristic 0. However, in proving part (i), it will not be necessary to consider those cases separately.

To begin the proof of part (i), let I be a semiprime ideal of R such that $\sigma(I) = I$. Since δ is algebraic over F, there exist $n \in \mathbb{N}$ and $\alpha_i \in F$ such that

$$\delta^n(r) = \alpha_{n-1}\delta^{n-1}(r) + \cdots + \alpha_1\delta(r) + \alpha_0 r,$$

for all $r \in R$. Since $\sigma(I) = I$, it follows that if $0 < j < n$, we have $\delta^j(I^n) \subseteq I$. In light of the equation above, we now have $\delta^n(I^n) \subseteq I$.

Since $(n!)_q \neq 0$, applying Lemma 4(iii), we have $(\delta(I))^n \subseteq I$. Using Lemma 3(i), we see that $I+\delta(I)$ is an ideal of R such that $(I+\delta(I))^n \subseteq I$. Since I is a semiprime ideal, we know that $I+\delta(I) \subseteq I$, which immediately implies that $\delta(I) \subseteq I$. Thus I is δ-stable. □

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Department of Mathematics, DePaul University, 2320 N. Kenmore Avenue, Chicago, Illinois 60614, USA
E-mail address: jbergen@depaul.edu

Faculty of Computer Science, Białystok University of Technology, Wiejska 45A, 15-351 Białystok, Poland
E-mail address: piotrgr@pb.edu.pl